An Interesting Diophantine Equation

by George Greaves and John F. Rigby,
School of Mathematics, University of Wales College of Cardiff

The following problem was being discussed in the mathematics departments of various British universities a few years ago; we are not sure where it originated. The solution is of sufficient interest to be worth writing a short note about. The idea behind the solution was provided by George Greaves, and was then expanded into its present form by John Rigby.

Problem: Let x and y be positive integers such that

$$\frac{x^2 + y^2}{xy + 1} = n, \qquad (1)$$

where n is an integer. Show that n is a perfect square.

Editor's Note: Try the problem yourself before reading on!

As well as solving the original problem, we shall find all solutions of (1). It is convenient to split up the solution into a number of small stages.

Lemma 1. *If n=1, the only positive solution of (1) is x = y = 1.*

If $n = 1$, (1) can be written as $(y - x)^2 = 1 - xy$; the left-hand side is nonnegative, whilst the right-hand side is negative unless $x = y = 1$.

We shall now assume, unless otherwise stated, that (a,b) is a positive integral solution of (1) for a particular value of n ($n > 1$), with $b \geq a$, so that

$$a^2 + b^2 = (ab + 1)n. \qquad (2)$$

Lemma 2. $b > (n - 1)a$.

From (2), $b[b - (n - 1)a] = n + a(b - a) > 0$, so $b > (n - 1)a$.

Corollary. $b > a$.

Lemma 3. $b^2 + (nb - a)^2 = [b(nb - a) + 1]n$.

This is just a rewriting of (2). Lemma 3 shows that if (a,b) is a solution of (1), then so is $(b,nb - a)$; and $b > a$, $nb - a = (n - 1)b + (b - a) > b$, so $(b,nb - a)$ is a strictly greater solution than (a,b).

This lemma provides a method of obtaining a sequence of strictly increasing solutions of (1) when $n = k^2$ ($k > 1$), starting with the solution (k,k^3). For instance when $k = 2$ we have (2,8), (8,30), (30,112), ... The solution (k,k^3) was originally found by trial and error, but its significance will appear later.

Lemma 4. $(na - b)^2 + a^2 = [(na - b)a + 1]n.$ (3)

This is just Lemma 3 with a and b interchanged, and is again a direct consequence of (2). If $na - b < 0$ then the right-hand side of (3) is nonpositive whilst the left-hand side is positive since $a > 0$; this is a contradiction. Hence $na - b \geq 0$. Also $na - b < a$ by Lemma 2, and $a < b$. Hence $(na - b, a)$ is a positive solution of (1), strictly less than (a,b) unless $na - b = 0$.

If we start with any positive solution (a,b) of (1) for a particular value of n, Lemma 4 provides a method of obtaining a sequence of strictly decreasing positive solutions; this is the exact reverse of the previous method of obtaining a strictly increasing sequence. Such a sequence cannot be infinite, so we must reach a positive solution (c,d) in the sequence such that $nc - d = 0$. Then, from Lemma 4 with (a,b) replaced by (c,d), we deduce that $c^2 = n$. Hence n is a perfect square, which is what we required to prove, and since $c^2 + d^2 = (cd + 1)n$ we see that $d = c^3$.

We have seen that, starting with any positive solution of (1), the decreasing sequence of solutions eventually leads us to the solution (c,c^3) with $n = c^2$; hence every solution with $n = c^2$ occurs in the increasing sequence starting from (c,c^3).

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