My Favorite Proof:
Sylvester's Problem

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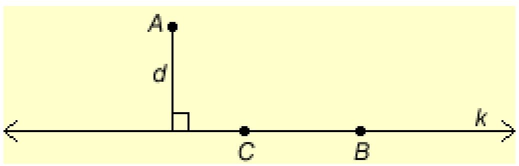
Sylvester's Problem is to prove that "It is not possible to arrange a finite number of points so that a line through every two of them passes through a third unless they are all on a single line." If you try to draw a bunch of dots that would disprove this statement, I think you'll quickly be convinced that the statement is true. However, actually proving the statement isn't quite as easy. In fact, Sylvester himself, a very great mathematician, never proved it.

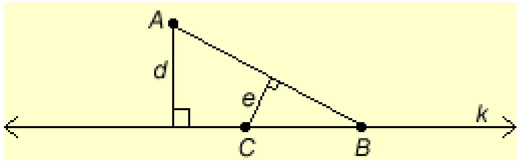
This problem stumped the entire mathematical community for forty years, from 1893 when it was first proposed until 1933 when it was finally solved by T. Gallai. Gallai's proof was extremely complicated and accessible to only a fraction of mathematicians. However, in 1948 L. M. Kelly published the following proof of Sylvester's Problem, a proof that is extremely simple considering the amount of time that the problem went unsolved.

Assume the given statement is false. In other words, assume there is some way to draw a bunch of points on a piece of paper so that any line that passes through two of these points also passes through a third. Now think about the distances between the points in our diagram and the lines. For every point, write down its distance to every line that it's not on. Make a huge list with the distance between every point and every line, and take the smallest distance on this list. This represents the distance between the point and the line in our diagram which are closest together.

Let's call the point *A* and the line *k*, and the shortest segment between them *d*. It is very important to realize that no point can possibly be nearer than the length of *d* to any line, because *d* is the shortest distance between any point and any line in our diagram. Since we assumed for the sake of argument that every line between two points in our diagram passes through a third point, we know that line *k* has at least three of the points in our diagram on it. At least two of these points will be on the same side of the line segment *d*.



Call the point that's further from line segment *d* "*B*" and the one that's closer "*C*". Now let's draw line *AB*. While we're at it, lets also look at the distance between *C* and line *AB*. Let's call it *e*.



Oh no! Segment *e* is shorter than *d*, the distance that we decided was the absolute minimum distance between a point and a line in our diagram! That's impossible! This is a contradiction of our assumption! This means what we were trying to prove in the first place was true after all, and there is no way to draw a bunch of points on a piece of paper so that any line that passes through two of these points also passes through a third, unless all of the points are on one straight line.

Resources:

Engel, Arthur. *Problem Solving Strategies*. New York: Springer-Verlag, 1998.

Singh, Simon. *Fermat's Last Theorem*. London: Fourth Estate, 1997.

Suggested further reading on unsolved problems:

Clawson, Calvin. *Mathematical Mysteries*. Cambridge: Perseus Books, 1996.

Singh, Simon. *Fermat's Last Theorem*. London: Fourth Estate, 1997.