2008 – 2009 Log1 Contest Round 3
Theta Individual

Name: ___________________

4 points each		
1	What is the sum of the first 5 Fibonacci numbers if the first two are 1, 1?	
2	If two cards are drawn from a standard 52 card deck, what is the probability of drawing a red king and a black queen?	
3	How many regular polygons can tessellate a plane?	
4	A cube of side length 2 has its side lengths increased by a factor of 3. By how much has the surface area increased?	
5	What is the volume of a right circular cylinder with radius 2π and height 2009?	

5 points each		
6	Stacey decides to add the numbers from 1 to her favorite number. If she gets a sum of 447, and realizes that she forgot to add one number, what is Stacey's favorite number?	
7	How many integer factors does 2009 have?	
8	What is the probability that I get 5 heads when I flip 8 fair coins?	
9	If $a+b=\frac{5}{6}$ and $a^2+b^2=\frac{13}{36}$, then what is the value of a^3+b^3?	
10	Using 5 chords to cut a circle, what is the maximum number of pieces that can be made?	

6 points each		
11	An equilateral triangle is inscribed in a circle that is inscribed in another equilateral triangle. If a point is selected at random inside the larger triangle what is the probability that the point will lie inside the circle but outside the smaller triangle?	
12	What is the remainder when $x^5+3x^4-4x^3+2x^2-5x+1$ is divided by $x+1$?	
13	What is the greatest common factor of 3397 and 2449?	
14	How many integer values of n exist such that $2<\sqrt[n]{2009}<10$?	
15	What is the total distance travelled by a ball that rebounds to $\frac{6}{7}$ of its drop height when dropped from a 2009 foot building?	

2008 – 2009 Log1 Contest Round 3
Alpha Individual

Name: ___________________

4 points each		
1	What is the sum of the first 5 Fibonacci numbers if the first two are 1, 1?	
2	If two cards are drawn from a standard 52 card deck, what is the probability of drawing a red king and a black queen?	
3	How many regular polygons can tessellate a plane?	
4	A cube of side length 2 has its side lengths increased by a factor of 3. By how much has the surface area increased?	
5	What is the smallest positive value of x in radians such that the expression $\sin\left(\cos^{-1}(\tan x)\right)$ equals one?	

5 points each		
6	Stacey decides to add the numbers from 1 to her favorite number. If she gets a sum of 447, and realizes that she forgot to add one number, what is Stacey's favorite number?	
7	How many integer factors does 2009 have?	
8	What is the probability that I get 5 heads when I flip 8 fair coins?	
9	If $a+b=\frac{5}{6}$ and $a^2+b^2=\frac{13}{36}$, then what is the value of a^3+b^3?	
10	3 sides of a triangle are part of an arithmetic sequence, if all sides are integers between 7 and 15, inclusive, then what is the probability that the triangle is obtuse?	

6 points each		
11	An equilateral triangle is inscribed in a circle that is inscribed in another equilateral triangle. If a point is selected at random inside the larger triangle what is the probability that the point will lie inside the circle but outside the smaller triangle?	
12	What is the remainder when $x^5+3x^4-4x^3+2x^2-5x+1$ is divided by $x+1$?	
13	What is the greatest common factor of 3397 and 2449?	
14	How many integer values of n exist such that $2<\sqrt[n]{2009}<10$?	
15	If the vector $<4,a,b>$ is orthogonal to $<3,2,4>$ and $<-1,1,-3>$, what is $a+b$?	

2008 – 2009 Log1 Contest Round 3
Mu Individual

Name: ___________________

	4 points each	
1	What is the sum of the first 5 Fibonacci numbers if the first two are 1, 1?	
2	If two cards are drawn from a standard 52 card deck, what is the probability of drawing a red king and a black queen?	
3	How many regular polygons can tessellate a plane?	
4	Evaluate: $\lim_{x\to\infty}\left(1+\frac{1}{e^x}\right)^{e^x}$	
5	What is the smallest positive value of x in radians such that the expression $\sin\left(\cos^{-1}(\tan x)\right)$ equals one?	

	5 points each	
6	Stacey decides to add the numbers from 1 to her favorite number. If she gets a sum of 447, and realizes that she forgot to add one number, what is Stacey's favorite number?	
7	How many integer factors does 2009 have?	
8	What is the probability that I get 5 heads when I flip 8 fair coins?	
9	What is the probability that when I integrate the function $f(x)=\cos x$ over an interval of width π, that I obtain a positive value?	
10	3 sides of a triangle are part of an arithmetic sequence, if all sides are integers between 7 and 15, inclusive, then what is the probability that the triangle is obtuse?	

	6 points each	
11	An equilateral triangle is inscribed in a circle that is inscribed in another equilateral triangle. If a point is selected at random inside the larger triangle what is the probability that the point will lie inside the circle but outside the smaller triangle?	
12	What is the remainder when $x^5+3x^4-4x^3+2x^2-5x+1$ is divided by $x+1$?	
13	What is the greatest common factor of 3397 and 2449?	
14	If $y=3$ when $x=0$ and $yy'=3x^2$, then what is $\lvert y\rvert$ when $x=2$?	
15	If the vector $<4,a,b>$ is orthogonal to $<3,2,4>$ and $<-1,1,-3>$, what is $a+b$?	

2008 – 2009 Log1 Contest Round 3
Individual Answers

#	Theta Answers	Alpha Answers	Mu Answers
1	12	12	12
2	$\frac{2}{663}$	$\frac{2}{663}$	$\frac{2}{663}$
3	3	3	3
4	192	192	e
5	$8036\pi^3$	π	π
6	30	30	30
7	12	12	12
8	$\frac{7}{32}$	$\frac{7}{32}$	$\frac{7}{32}$
9	$\frac{35}{216}$	$\frac{35}{216}$	$\frac{1}{2}$
10	16	$\frac{3}{16}$	$\frac{3}{16}$
11	$\frac{4\pi\sqrt{3}-9}{36}$	$\frac{4\pi\sqrt{3}-9}{36}$	$\frac{4\pi\sqrt{3}-9}{36}$
12	14	14	14
13	79	79	79
14	7	7	5
15	26117	-4	-4

2008 – 2009 Log1 Contest Round 3
Individual Solutions

Th	Al	Mu	Solution
1	1	1	The first five numbers of the Fibonacci sequence are 1, 1, 2, 3, and 5. The sum of these numbers is 12.
2	2	2	The probability of drawing a red king and black queen is the sum of the probability of drawing a red king and then a black queen and the probability of drawing a black queen and then a red king: $\left(\frac{2}{52}\right)\left(\frac{2}{51}\right)+\left(\frac{2}{52}\right)\left(\frac{2}{51}\right)=\left(\frac{8}{2652}\right)=\frac{2}{663}$. Or, we can simply take the number of ways of drawing a red king (2) times the number of ways of drawing a black queen (2) and dividing by the number of ways of choosing 2 cards; 52C2.
3	3	3	In order for a regular polygon to tessellate a plane one of its interior angles must be a factor of 360. There are only 3 regular polygons that have this attribute: a triangle, a quadrilateral, and a hexagon.
4	4		The surface area of a cube can be expressed as $6e^2$, where e is the side length of the cube. Thus: $\Delta surfacearea = 6(3e)^2 - 6e^2 = 48e^2 = 48(2)^2 = 192$
		4	This should look familiar as an altered form of the definition of e, $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$. Since ln(x) is continuous, we can take logs and take the limit outside. We can then write the problem as: $\exp\left(\lim_{x\to\infty}\frac{\ln(1+e^{-x})}{e^{-x}}\right)$,this can be solved using L'Hopitals rule to get exp(1)=e.
5			The volume of a right cylinder can be determined by the expression: $\pi r^2 h$. Plug in the values: $\pi(2\pi)^2(2009) = 8036\pi^3$
	5	5	Since the maximum for the sine function in the expression is 1, we set the expression equal to 1 and solve for x. $\sin\left(\cos^{-1}(\tan x)\right)=1$ $\cos^{-1}(\tan x)) = \sin^{-1}(1) = \frac{\pi}{2}$ $\tan x = \cos\left(\frac{\pi}{2}\right) = 0$ $x = \tan^{-1}(0) = 0$ But since 0 is not positive, the next value for which $\tan x = 0$ is π.
6	6	6	This problem can be expressed by the equation: $\frac{n(n+1)}{2} = 447 + k$, where n is Stacey's favorite number and k is the number she skipped. Thus: $\frac{n(n+1)}{2} = 447 + k$ $n^2 + n = 894 + 2k$ Since $n^2 + n$ is close to n^2, we find the smallest perfect square larger than 894. This happens to be 900. Thus Stacey's favorite number is 30.

7	7	7	Prime factorization of 2009: $2009 = 7^2 \cdot 41^1$ Thus the total number of *positive* factors is $3 \times 2 = 6$, yet to account for *negative* factors as well double that value. 12.
8	8	8	Binomial Probability: $\binom{8}{5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^3 = (56)\left(\frac{1}{256}\right) = \frac{7}{32}$
9	9		$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$ $a^2 + b^2 = \frac{13}{36}$ $a + b = \frac{5}{6}$ $\therefore (a+b)^2 = \left(\frac{5}{6}\right)^2$ $a^2 + 2ab + b^2 = \frac{25}{36}$ $(a^2 + b^2) + 2ab = \frac{25}{36}$ $\left(\frac{13}{36}\right) + 2ab = \frac{25}{36}$ $\therefore ab = \frac{6}{36}$ $a^3 + b^3 = (a+b)(a^2 + b^2 - ab)$ $= \left(\frac{5}{6}\right)\left(\frac{13}{36} - \frac{6}{36}\right)$ $= \left(\frac{5}{6}\right)\left(\frac{7}{36}\right) = \frac{35}{216}$
		9	$\int_a^{a+\pi} \cos x dx = \sin x \|_a^{a+\pi}$ $= \sin(a+\pi) - \sin(a) = \sin(a)\cos(\pi) + \sin(\pi)\cos(a) - \sin(a) = -2\sin(a)$ This will be positive whenever sin(a) is negative which is 1/2 the time..
10			The first chord cuts the circle into two pieces, the second (intersecting the first) adds 2 more for a total of 4. The third (intersecting both previous chords but not at their intersection) adds 3, so 7 total. The 4th adds 4 more and the 5th, 5 more for a total of 16 pieces.
	10	10	There exist 16 triangles that have these described attributes (9-8-7, 10-9-8, 11-10-9, 11-9-7, 12-11-10, 12-10-8, 13-12-11, 13-11-9, 13-10-7, 14-13-12, 14-12-10, 14-11-8, 15-14-13, 15-13-11, 15-12-9 and 15-11-7. Using the law of cosines, for the largest angle to be obtuse, we must have $a^2 + b^2 < c^2$. This only happens in the triangles: 13-10-7, 14-11-8 and 15-11-7.
11	11	11	The center of the circle will also be the centroid of both triangles. The distance from the centroid to the far vertex is 2/3 the median (and altitude in an equilateral triangle). If we left 2s equal the radius of the circle, then 3s will be the height of the small triangle and 6s the height of the larger triangle. Then, $2\sqrt{3}s$ will be the side length of the smaller triangle and $4\sqrt{3}s$ the side length of the larger triangle. The desired probability is then the (Area of the circle minus the area of the smaller triangle) divided by the area of the larger triangle. $\frac{4\pi\sqrt{3} - 9}{36}$

12	12	12	The remainder can be found by division (synthetic or otherwise) or by the remainder theorem which states the remainder when P(x) is divided by (x-a) is P(a). So, using x=-1, we get a remainder of 14.
13	13	13	Since 3397 = 2449 + 948, and number that divides 3397 and 2449 must divide 948, therefore the gcf of 3397 and 2449 is the same as 2449 and 948. Now 2449 = 2(948)+553 so all we need is the gcf of 948 and 553. Continuing this equals the gcf of 553 and 395 which equals the gcf of 395 and 158 which equals the gcf of 158 and 79 which equals 79. This methods is due to Euler.
14	14		By raising each number to the power n and then comparing the values, we see that the inequality only works for values of n between 4 and 10, inclusive. Thus 7.
		14	This equation can be solved by separating the x and y terms. $yy' = 3x^2$ $y\frac{dy}{dx} = 3x^2$ $ydy = \left(3x^2\right)dx$ $\int ydy = \int 3x^2 dx$ $\frac{1}{2}y^2 = x^3 + C$ $y^2 = 2x^3 + C$ $y = 3, x = 0, \therefore C = 9$ $y^2 = 2(2^3) + 9 = 25$ $\lvert y \rvert = 5$
15			The total distance is the sum of the distance the ball travels down and the distance the ball travels up; both of which are infinite geometric sequences with a common ratio of $\frac{6}{7}$. The downward distance has a first term of 2009 while the upward distance has a first term equal to the downward sequence's second term, 1722. $\frac{2009}{\left(1-\frac{6}{7}\right)} + \frac{1722}{\left(1-\frac{6}{7}\right)} = (7)(2009 + 1722) = 26117$
	15	15	Two vectors are orthogonal or perpendicular if their dot product is 0. So we have $4(3) + 2a + 4b = 0$ $4(-1) + a - 3b = 0$ Solving for a and b gives a=b=-2, so a+b=-4.

Alpha and Mu #10 as originally stated.

3 sides of a triangle are part of an arithmetic sequence, if all sides are between 7 and 15, inclusive, then what is the probability that the triangle is obtuse? No restrictions that the sides be integers.

This is much harder as the sides can be any numbers between 7 and 15. Let the three sides be:
$x - y, x, x + y$ subject to the conditions:
$x \geq 7, y \geq 0,$
$x - y \geq 7, and$
$x + y \leq 15$

Plotted; these conditions gives a feasible region bounded by the triangle with vertices (7,0), (15,0) and (11,4). This happens to be a right triangle with area 16 (the denominator of our probability). The condition that the triangle be obtuse means that $(x-y)^2 + x^2 < (x+y)^2$. Simplifying we get $x(x-4y) < 0$. Since x cannot be less than 0; $x - 4y < 0$; another half plane. This intersects the feasible region in the right triangle $\left(\frac{28}{3}, \frac{7}{3}\right), (12,3), (11,4)$; with area 5/3. Any point in this last triangle corresponds to an obtuse triangle whose side form an arithmetic sequence.

The desired probability is then (5/3)/16 = 5/48.