1. D
2. A
3. B
4. A
5. D
6. D
7. C
8. C
9. D
10. E
11. B
12. C
13. A
14. C
15. A
16. B
17. C
18. D
19. B
20. B
21. D
22. D
23. C
24. A
25. D
26. A
27. C
28. A
29. B
30. E
Answers: DABAD DCCDE BCACA BCDBB DDCA D ACBE

1) A linear combination of $y = \sin ax$ and $y = \cos ax$ has period $\frac{2\pi}{a}$, which here is $\frac{\pi}{3}$. The amplitude of $y = m \sin ax + n \cos ax$ is $\sqrt{m^2 + n^2}$, which here is 13.

2) $\theta$ is in the first quadrant, while $\phi$ is in the second quadrant. Therefore, $\sin \theta = \frac{5}{13}$, $\sin \phi = \frac{4}{5}$, and $\cos \phi = -\frac{3}{5}$. $\cos(\phi - \theta) = \cos \phi \cos \theta + \sin \phi \sin \theta = -\frac{12}{13} \cdot \frac{3}{5} + \frac{5}{13} \cdot \frac{4}{5} = -\frac{36}{65} + \frac{20}{65} = -\frac{16}{65}$, so $\sec(\phi - \theta) = -\frac{65}{16}$.

3) The value of $\sin \theta$ oscillates between 1 and -1 with a period of $2\pi$, but cosine is an even function, where $\cos(-\theta) = \cos \theta$. Therefore, $\cos(\sin \theta) = \cos(\lfloor \sin \theta \rfloor)$. $|\sin \theta|$ oscillates between 0 and 1 with a period of $\pi$, so the period of $\cos(\sin \theta)$ is $\pi$.

4) Converting to sine and cosine, the equation becomes $\frac{\sin \theta}{\cos \theta} = \frac{2\sin \theta}{2\cos \theta}$, or $\frac{\sin \theta}{\cos \theta} = \frac{\cos \theta}{2\cos \theta} \cdot \frac{\theta}{\theta}$ is bijective over the interval, so $\sin \theta = \cos \theta$, so the 2 solutions are $\theta = \frac{\pi}{4}$ and $\theta = \frac{5\pi}{4}$.

5) The smallest angle in the triangle is opposite the shortest side. By the law of cosines, $\cos \theta = \frac{5^2 + 7^2 - 4^2}{2 \cdot 5 \cdot 7} = \frac{29}{35}$, so $\sin \theta = \frac{8\sqrt{6}}{35}$. $A + B + C = 49$.

6) $\frac{(b+c)-(a+b)}{22-21} = \frac{(c+a)-(a+b)}{23-22}$, so $c - a = a - b = \frac{c-b}{2}$. Since $\frac{a+b}{21} = a - b$, 10a = 11b. Since $\frac{b+c}{22} = \frac{c-b}{2}$, 5c = 6b. Since $\frac{c+a}{23} = c - a$, 11c = 12a. The sides of the triangle are therefore $a = 11t$, $b = 10t$, and $c = 12t$ for some constant $t$, so $B$ is the smallest angle. By the Law of Cosines, $\cos B = \frac{144t^2 + 121t^2 - 100t^2}{2 \cdot 12t \cdot 11t} = \frac{165t^2}{264t^2} = \frac{5}{8}$.

7) Taking the tangent of both sides, $\frac{a-x}{1+ax} = \frac{1}{7}$. Cross-multiplying, $7(a - x) = 1 + ax$, or $ax - 7a + 7x = -1$. Using Simon’s Favorite Factoring Trick, $(a + 7)(x - 7) = -50$. Both $a$ and $x$ are positive integers, so $x - 7$ must be negative; solving $x - 7 = \{-1, -2, -5\}$ gives the solution set $x = \{6, 5, 2\}$, which corresponds to $a = \{43, 18, 3\}$. The sum of the elements of this set is 64.

8) Recognize that the first three points all lie on the plane $x + y + z - 6 = 0$. Translate the triangle that the first three points form so that the first point is the origin and the triangle has endpoints at $(0, 0, 0)$, $(3, 0, -3)$, and $(2, 3, -5)$. The lengths of the vectors between the origin and the second two points are $3\sqrt{2}$ and $\sqrt{38}$ respectively, and the cosine of the angle between them is $\frac{\sqrt{38}}{3\sqrt{2}/\sqrt{38}} = \frac{\sqrt{2}}{\sqrt{2\sqrt{3}}}$.

9) $\sin \theta = \frac{x + \frac{\pi}{3} \pm \alpha}{3}$, $a = \pi \pm \alpha x = \frac{5\pi}{3} \pm \alpha$ for some $0 < \alpha < \frac{\pi}{3}$, and the sum of these solutions is $6\pi$.

10) Item III fails when $\cos(\theta_1 + \theta_2) = -\frac{1}{\sqrt{2}} = -\cos(\theta_1 - \theta_2)$, so $\cos(\theta_1 - \theta_2) = \frac{1}{\sqrt{2}}$; solving these equations gives $\theta_1 = \frac{3\pi}{4}$ and $\theta_2 = \frac{\pi}{2}$. Item I represents $\frac{\sin(\theta_1 + \theta_2)}{\cos(\theta_1 + \theta_2)} = \tan(\theta_1 + \theta_2)$, which after
setting $\theta_1 + \theta_2 = \frac{5\pi}{4}$ results in a value of 1 for all values of $\theta_1$ and $\theta_2$ that satisfy the conditions. For item II, note that tan $\theta_1 + \tan \theta_2 + \tan \theta_1 \tan \theta_2 = (1 + \tan \theta_1)(1 + \tan \theta_2) - 1 = (1 + \tan \theta_1 + \tan \theta_1 + \tan \theta_2)(1 + \tan \left(\frac{5\pi}{4} - \theta_1\right)) - 1$. The second multiplicand simplifies to $1 + \frac{1 - \tan \theta_1}{1 + \tan \theta_1} = \frac{2}{1 + \tan \theta_1}$, so the overall expression is $2 - 1 = 1$.

11) The expressions have a product of $60\sin^2 \theta + \cos^2 \theta = 60$. The positive integers with the smallest possible sum that multiply to 60 are 6 and 10, which add to 16.

12) The middle term is the average of the two outer terms; $\cot(\alpha - \beta) + \cot(\alpha + \beta) = 6\cot \alpha$. Using the addition formula for tangent and letting $A = \tan \alpha$ and $B = \tan \beta$.

Combining like terms, $\frac{2A(1+B^2)}{A^2-B^2} = \frac{6}{A}$, or $2A^2(1 + B^2) = 6(A^2 - B^2)$. This simplifies to $B^2 = \frac{2A^2}{A^2+3}$.

Converting the right side to sines yields $\frac{\sin^2 \beta}{\cos^2 \beta} = \frac{2}{\sin^2 \alpha + 3 \cos^2 \alpha} = \frac{2 \sin^2 \alpha}{1 + 2 \cos^2 \alpha}$. Cross multiplying gives $\sin^2 \beta (1 + 2 \cos^2 \alpha) = 2 \sin^2 \alpha \cos^2 \beta$, or $\sin^2 \beta (3 - 2 \sin^2 \alpha) = 2 \sin^2 \alpha - 2 \sin^2 \alpha \sin^2 \beta$. This simplifies to $3 \sin^2 \beta = 2 \sin^2 \alpha$, or $\frac{\sin^2 \alpha}{\sin^2 \beta} = \frac{3}{2}$.

13) Rearranging, $2 \cos \theta + 3 \cos^3 \theta = \sin^2 \theta$. Squaring, $4 \cos^2 \theta + 4 \cos^4 \theta + \cos^6 \theta = \sin^4 \theta$. Substituting $\cos^2 \theta = 1 - \sin^2 \theta$ yields $-\sin^6 \theta + 7 \sin^4 \theta - 15 \sin^2 \theta + 9 = \sin^4 \theta$, and separating yields $\sin^6 \theta = 6 \sin^4 \theta - 15 \sin^2 \theta + 9$. $a + b + c = 0$.

14) The slopes of these lines can be represented by the vectors $(1,2)$ and $(1,-7)$. The dot product gives $(1,2) \cdot (1,-7) = |(1,2)|| (1,-7)| \cos \theta_{\text{bet}}$. $1 - 14 = \sqrt{5}/\sqrt{50} \cos \theta_{\text{bet}}$, so $\cos \theta_{\text{bet}} = -\frac{13}{\sqrt{5}}$. This represents an obtuse angle; the cosine of the acute angle is $\frac{13}{\sqrt{50}}$ and $\sin \theta_{\text{ac}} = \frac{9}{\sqrt{50}}$.

15) Expanding out $\tan(\arctan z_1 + \arctan z_2 + \arctan z_3 + \arctan z_4)$ as $\tan((\arctan z_1 + \arctan z_2) + (\arctan z_3 + \arctan z_4))$ with the tangent addition formula results in the fraction $\frac{z_1 + z_2 + z_3 + z_4}{1 - z_1 z_2 - z_1 z_3 - z_2 z_3 - z_3 z_4}$. FOILing and simplifying results in $\frac{z_1 + z_2 + z_3 + z_4}{1 - z_1 z_2 - z_1 z_3 - z_2 z_3 - z_3 z_4}$, which by Vieta’s is equal to $\frac{e_1 - e_3}{1 - e_2 + e_4}$. Using the polynomial, this is $\frac{2}{1 - 0 + 2} = 2$.

16) The primitive period of $\sin \frac{x}{a} \cos \frac{x}{b}$ is $2\pi \text{LCM}(a,b)$. The primitive period of each term is $40\pi$ and $112\pi$ respectively. The primitive period of the sum of two untranslated sinusoidal functions with periods $a$ and $b$ is $\text{LCM}(a,b)$. Here, this is $\text{LCM}(40\pi, 112\pi) = 560\pi$.

17) Note that $\tan\left(\frac{\pi}{4} - \theta\right) = \frac{1 - \tan \theta}{1 + \tan \theta}$, so $\tan \theta = \frac{1 - \tan \left(\frac{\pi}{4} - \theta\right)}{1 + \tan \left(\frac{\pi}{4} - \theta\right)}$ and $\tan \theta + \tan \left(\frac{\pi}{4} - \theta\right) + \tan \theta \tan \left(\frac{\pi}{4} - \theta\right) + 1 = (1 + \tan \theta) \left(1 + \tan \left(\frac{\pi}{4} - \theta\right)\right) = 2$. The product is equal to $\prod_{n=0}^{45} (1 + \tan n\theta) (1 + \tan(90\theta - \theta)) = 2^{23}$, so $A + B = 25$.

18) A rose curve with an argument of $\theta$ being multiplied by an even integer $n$ has $2n$ petals. $2 \cdot 2022 = 4044$. 

Page 3 of 5
The reference angles of $\frac{1337\pi}{3}$ and $\frac{1337\pi}{6}$ are $\frac{5\pi}{3}$ and $\frac{5\pi}{6}$ respectively. In Cartesian, these points are $\left(\frac{3}{2}, -\frac{3\sqrt{3}}{2}\right)$ and $(-4\sqrt{3}, 4)$. Using the Distance Formula on these points gives a distance of $\sqrt{73 + 24\sqrt{3}}$. $A + H + S = 100$.

Converting to standard form, $r = \frac{3/4}{1+2\cos \theta}$. Plugging in $\theta = 0$ yields the point $\left(\frac{1}{4}, 0\right)$, and plugging in $\theta = \pi$ yields the point $\left(\frac{3}{4}, 0\right)$. The center of the conic is therefore at $\left(\frac{1}{2}, 0\right)$. One focus is at the origin, and the other focus is an equal distance away from the center at $(1,0)$. The other latus rectum passes through this point as part of the line $x = 1$.

The magnitude of $\sqrt{3} - 3i$ is $\sqrt{12}$, so $\sqrt{3} - 3i = \sqrt{12} \left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) = \sqrt{12} \cos \left(-\frac{\pi}{3}\right)$.

By De Moivre’s, this taken to the power of 413 is $12^{413/2} \cos \left(-\frac{413\pi}{3}\right) = 12^{206} \sqrt{12} \cos \left(\frac{\pi}{3}\right) = 12^{206}(\sqrt{3} + 3i)$.

Using the complex definition of sine, $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$, we are solving $\frac{e^{iz} - e^{-iz}}{2i} = i$, or $e^{iz} - e^{-iz} + 2 = 0$. Multiplying by $e^{iz}$ gives a quadratic in $e^{2iz}$. $e^{2iz} - 2e^{iz} - 1 = 0$. Substituting $u = e^{iz}$ and solving for $u$, if $u^2 - 2u - 1 = 0$ then $u = 1 \pm \sqrt{2}$. Testing $u = 1 + \sqrt{2}$ gives $z = i \ln(\sqrt{2} - 1)$, but this is negative and not in the principal value range of the complex arcsine. Testing $u = 1 - \sqrt{2}$ gives $z = -i \ln(1 - \sqrt{2}) = i \ln(1 + \sqrt{2})$, which is in the principal value range.

$\sin \theta + \cos \theta = \sqrt{2} \sin \left(\theta + \frac{\pi}{4}\right)$. A polar function in this form is a parabola only if the constant term equals the coefficient of the sine or cosine term (or its negative), so $a = \sqrt{2}$.

Let $S_n = a_n + ib_n$. Then $S_n = S_{n+1}S_{n-1}$. Similarly, $S_{n+1} = S_nS_{n+2} = S_{n+1}S_{n-1}S_{n+2}$, so $S_{n-1}S_{n+2} = 1$ and $S_{n-1} = \frac{1}{S_{n+2}}$. Index-shifting, $S_n = \frac{1}{S_{n+3}} = S_{n+6}$, so the sequence is periodic. $S_{02022} = S_0$, so $a_{02022} = 2, b_{02022} = 7$, and $a_{02022}^2 + b_{02022}^2 = 4 + 49 = 53$.

This equals $\left(5\sqrt{2} \cos \frac{\pi}{4}\right)^2(3 + i)^2 = (2500 \cos \pi)(9 + 6i + i^2) = -20000 - 15000i$. The sum of the digits of $A + B = 35000$ is 8.

By the sine addition formula, $r(\sin \theta \cos(\arctan 2) - \cos \theta \sin(\arctan 2)) = \frac{1}{\sqrt{5}}$ sin(\arctan 2) = $\frac{2}{\sqrt{5}}$ and cos(\arctan 2) = $\frac{1}{\sqrt{5}}$ so this simplifies to $r \cos \theta - 2r \sin \theta = 1$. Converting from polar, $f(x) - 2x = 1$, or $f(x) = 2x + 1$. $f(2) = 5$.

$\ln(-\sqrt{3} - i) = \ln(2) + \ln \left(-\frac{\sqrt{3}}{2} - \frac{i}{2}\right) = \ln(2) + \ln e^{-\pi/6} = \ln(2) - \frac{5\pi}{6}$. $A + B + C = 13$. Note that the imaginary part of the principal value of $\ln(z)$ always lies in the range $(-\pi, \pi]$.

This is an ellipse with foci at $(0, -i)$ and $(0, i)$ with major axis length 4. The semimajor axis has length 2, and the focal radius is 1. Solving $1^2 = 2^2 - x^2$ gives a semiminor axis of length $\sqrt{3}$. Thus, the area of the ellipse is $2\pi \sqrt{3}$.

The determinant is $\cos \frac{\pi}{4} \cdot \cos \frac{5\pi}{4} \cdot \cos \frac{\pi}{2} = \cos 2\pi = 1$. This is an easy triangular matrix to invert.
\[
\begin{bmatrix}
\text{cis} \pi/4 & \text{cis} 3\pi/2 & \text{cis} \pi/2 \\
0 & \text{cis} 5\pi/4 & \text{cis} \pi/4 \\
0 & 0 & \text{cis} \pi/2
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\] 
\(\Rightarrow R_1 \div= \text{cis} \pi/4\)

\[
\begin{bmatrix}
1 & \text{cis} 5\pi/4 & \text{cis} \pi/4 \\
0 & \text{cis} 5\pi/4 & \text{cis} \pi/4 \\
0 & 0 & \text{cis} \pi/2
\end{bmatrix}
\begin{bmatrix}
\text{cis} -\pi/4 & -1 & 0 \\
0 & \text{cis} 3\pi/4 & 0 \\
0 & 0 & 1
\end{bmatrix}
\] 
\(\Rightarrow R_1 = R_2\)

\[
\begin{bmatrix}
1 & \text{cis} 5\pi/4 & \text{cis} \pi/4 \\
0 & \text{cis} 5\pi/4 & \text{cis} \pi/4 \\
0 & 0 & \text{cis} \pi/2
\end{bmatrix}
\begin{bmatrix}
\text{cis} 0 & -1 & 0 \\
0 & \text{cis} \pi/4 & 0 \\
0 & 0 & 1
\end{bmatrix}
\] 
\(\Rightarrow R_2 \times= \text{cis} 3\pi/4\)

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & \text{cis} \pi \\
0 & 0 & \text{cis} \pi/2
\end{bmatrix}
\begin{bmatrix}
\text{cis} -\pi/4 & -1 & 0 \\
0 & \text{cis} 3\pi/4 & 0 \\
0 & 0 & 1
\end{bmatrix}
\] 
\(\Rightarrow\) Convert left matrix to rectangular

\[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\text{cis} -\pi/4 & -1 & 0 \\
0 & \text{cis} 3\pi/4 & 0 \\
0 & 0 & 1
\end{bmatrix}
\] 
\(\Rightarrow R_2 = iR_3\)

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\text{cis} -\pi/4 & -1 & 0 \\
0 & \text{cis} 3\pi/4 & 0 \\
0 & 0 & 1
\end{bmatrix}
\] 
\(\Rightarrow R_3 \div= i\)

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\text{cis} -\pi/4 & -1 & 0 \\
0 & \text{cis} 3\pi/4 & 0 \\
0 & 0 & 1
\end{bmatrix}
\] 
\(\Rightarrow\) Convert right matrix to rectangular

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\text{cis} -\pi/4 & -1 & 0 \\
0 & \text{cis} 3\pi/4 & 0 \\
0 & 0 & 1
\end{bmatrix}
\] 
\(\Rightarrow R_3 \div= i\)

The sum of the entries in the inverse matrix is \(-1 - 2i\). \(A + B = 3\).

\[\text{The areas of triangle } OPR, \text{ sector } POQ, \text{ and triangle } OPS \text{ are } \sin \theta/2, \theta/2, \text{ and } \tan \theta/2 \text{ respectively. For all values in the range } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \text{ except at } 0, \text{ these are ordered, so } \frac{\sin \theta}{2} < \frac{\theta}{2} < \frac{\tan \theta}{2}. \text{ Multiplying by } 2 \text{ and dividing by } \sin \theta \text{ (note that } \theta \text{ is always positive for area), this becomes } 1 < \frac{\theta}{\sin \theta} < \sec \theta. \text{ Inverting, } 1 > \frac{\sin \theta}{\theta} > \cos \theta. \text{ As } \theta \text{ becomes close to } 0, \cos \theta \text{ approaches } 1, \text{ and } \frac{\sin \theta}{\theta} \text{ becomes sandwiched between a value approaching } 1 \text{ and } 1. \text{ Thus, } \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1. \text{ Therefore, } \lim_{\theta \to 0} \frac{\sqrt{1 - \cos 2\theta}}{4\theta} = \frac{\sqrt{2} \sin \theta}{4\theta}, \text{ so } \lim_{\theta \to 0} \frac{\sqrt{1 - \cos 2\theta}}{4\theta} = \lim_{\theta \to 0} \frac{\sqrt{2} \sin \theta}{4\theta} = \frac{\sqrt{2}}{4}.\]