1. D
2. A
3. B
4. A
5. D
6. D
7. C
8. C
9. D
10. E
11. B
12. C
13. A
14. C
15. A
16. B
17. C
18. D
19. B
20. B
21. D
22. D
23. C
24. A
25. D
26. A
27. C
28. A
29. B
30. E

Answers: DABAD DCCDE BCACA BCDBB DDCAD ACABE

01) A linear combination of of $y = \sin ax$ and $y = \cos ax$ has period $\frac{2\pi}{a}$, which here is $\frac{\pi}{3}$. The amplitude of $y = m \sin ax + n \cos ax$ is $\sqrt{m^2 + n^2}$, which here is 13.

02) θ is in the first quadrant, while ϕ is in the second quadrant. Therefore, $\sin\theta = \frac{5}{13}$, $\sin\phi = \frac{4}{5}$, and $\cos\phi = -\frac{3}{5}$. $\cos(\phi - \theta) = \cos\phi\cos\theta + \sin\phi\sin\theta = -\frac{12}{13}\cdot\frac{3}{5} + \frac{5}{13}\cdot\frac{4}{5} = -\frac{36}{65} + \frac{20}{65} = -\frac{16}{65}$, so $\sec(\phi - \theta) = -\frac{65}{16}$.

03) The value of $\sin\theta$ oscillates between 1 and -1 with a period of 2π, but cosine is an even function, where $\cos(-\theta) = \cos\theta$. Therefore, $\cos(\sin\theta) = \cos(|\sin\theta|)$. $|\sin\theta|$ oscillates between 0 and 1 with a period of π, so the period of $\cos(\sin\theta)$ is π.

04) Converting to sine and cosine, the equation becomes $\frac{\sin\theta}{\cos\theta} = \frac{2^{\sin\theta}}{2^{\cos\theta}}$, or $\frac{\sin\theta}{2^{\sin\theta}} = \frac{\cos\theta}{2^{\cos\theta}}$. $\frac{\theta}{2^\theta}$ is bijective over the interval, so $\sin\theta = \cos\theta$, so the 2 solutions are $\theta = \frac{\pi}{4}$ and $\theta = \frac{5\pi}{4}$.

05) The smallest angle in the triangle is opposite the shortest side. By the law of cosines, $\cos\theta = \frac{5^2+7^2-4^2}{2\cdot5\cdot7} = \frac{29}{35}$, so $\sin\theta = \frac{8\sqrt{6}}{35}$. $A + B + C = 49$.

06) $\frac{(b+c)-(a+b)}{22-21} = \frac{(c+a)-(b+c)}{23-22} = \frac{(c+a)-(a+b)}{23-21}$, so $c - a = a - b = \frac{c-b}{2}$. Since $\frac{a+b}{21} = a - b$, $10a = 11b$. Since $\frac{b+c}{22} = \frac{c-b}{2}$, $5c = 6b$. Since $\frac{c+a}{23} = c - a$, $11c = 12a$. The sides of the triangle are therefore $a = 11t$, $b = 10t$, and $c = 12t$ for some constant t, so B is the smallest angle. By the Law of Cosines, $\cos B = \frac{144t^2+121t^2-100t^2}{2\cdot12t\cdot11t} = \frac{165t^2}{264t^2} = \frac{5}{8}$.

07) Taking the tangent of both sides, $\frac{a-x}{1+ax} = \frac{1}{7}$. Cross-multiplying, $7(a - x) = 1 + ax$, or $ax - 7a + 7x = -1$. Using Simon's Favorite Factoring Trick, $(a + 7)(x - 7) = -50$. Both a and x are positive integers, so $x - 7$ must be negative; solving $x - 7 = \{-1, -2, -5\}$ gives the solution set $x = \{6,5,2\}$, which corresponds to $a = \{43,18,3\}$. The sum of the elements of this set is 64.

08) Recognize that the first three points all lie on the plane $x + y + z - 6 = 0$. Translate the triangle that the first three points form so that the first point is the origin and the triangle has endpoints at $(0,0,0)$, $(3,0,-3)$, and $(2,3,-5)$. The lengths of the vectors between the origin and the second two points are $3\sqrt{2}$ and $\sqrt{38}$ respectively, and the cosine of the angle between them is $\frac{\langle 3,0,-3\rangle\cdot\langle 2,3,-5\rangle}{3\sqrt{2}\sqrt{38}} = \frac{7}{2\sqrt{19}}$. The sine of the angle between them is $\frac{3\sqrt{3}}{2\sqrt{19}}$, and the area of the triangle (which is the base of the tetrahedron) is $\frac{3\sqrt{3}}{2\cdot2\sqrt{19}}\cdot 3\sqrt{2}\sqrt{38} = \frac{9\sqrt{3}}{2}$. The distance from this triangle on the plane to the point $(-1,6,9)$ is $\frac{-1+6+9-6}{\sqrt{1+1+1}} = \frac{8}{\sqrt{3}}$, the height of the pyramid. The volume is $\frac{1}{3}\cdot\frac{9\sqrt{3}}{2}\cdot\frac{8}{\sqrt{3}} = 12$.

09) $1 - 4\sin^2 x = (4 - 4\sin^2 x) - 3 = 4\cos^2 x - 3$, so the LHS is equal to $4\cos^3 x - 3\cos x$, or $4\cos 3x$. By symmetry of cosine, the equation has solutions at $x = \frac{\pi}{3} \pm \alpha$, a $x = \pi \pm \alpha$ $x = \frac{5\pi}{3} \pm \alpha$ for some $0 < \alpha < \frac{\pi}{3}$, and the sum of these solutions is 6π.

10) Item III fails when $\cos(\theta_1 + \theta_2) = -\frac{1}{\sqrt{2}} = -\cos(\theta_1 - \theta_2)$, so $\cos(\theta_1 - \theta_2) = \frac{1}{\sqrt{2}}$; solving these equations gives $\theta_1 = \frac{3\pi}{4}$ and $\theta_2 = \frac{\pi}{2}$. Item I represents $\frac{\sin(\theta_1+\theta_2)}{\cos(\theta_1+\theta_2)} = \tan(\theta_1 + \theta_2)$, which after

setting $\theta_1 + \theta_2 = \frac{5\pi}{4}$ results in a value of 1 for all values of θ_1 and θ_2 that satisfy the conditions. For item II, note that $\tan\theta_1 + \tan\theta_2 + \tan\theta_1\tan\theta_2 = (1+\tan\theta_1)(1+\tan\theta_2) - 1 = (1+\tan\theta_1)\left(1+\tan\left(\frac{5\pi}{4}-\theta_1\right)\right) - 1$. The second multiplicand simplifies to $1 + \frac{1-\tan\theta_1}{1+\tan\theta_1} = \frac{2}{1+\tan\theta_1}$, so the overall expression is $2 - 1 = 1$.

11) The expressions have a product of $60^{\sin^2\theta+\cos^2\theta} = 60$. The positive integers with the smallest possible sum that multiply to 60 are 6 and 10, which add to 16.

12) The middle term is the average of the two outer terms; $\cot(\alpha-\beta) + \cot(\alpha+\beta) = 6\cot\alpha$. Using the addition formula for tangent and letting $A = \tan\alpha$ and $B = \tan\beta$, $\frac{1+AB}{A-B} + \frac{1-AB}{A+B} = \frac{6}{A}$. Combining like terms, $\frac{2A(1+B^2)}{A^2-B^2} = \frac{6}{A}$, or $2A^2(1+B^2) = 6(A^2-B^2)$. This simplifies to $B^2 = \frac{2A^2}{A^2+3}$. Converting the right side to sines yields $\frac{\sin^2\beta}{\cos^2\beta} = \frac{2}{\sin^2\alpha+3\cos^2\alpha} = \frac{2\sin^2\alpha}{1+2\cos^2\alpha}$. Cross multiplying gives $\sin^2\beta\,(1+2\cos^2\alpha) = 2\sin^2\alpha\cos^2\beta$, or $\sin^2\beta\,(3-2\sin^2\alpha) = 2\sin^2\alpha - 2\sin^2\alpha\sin^2\beta$. This simplifies to $3\sin^2\beta = 2\sin^2\alpha$, or $\frac{\sin^2\alpha}{\sin^2\beta} = \frac{3}{2}$.

13) Rearranging, $2\cos\aleph + \cos^3\aleph = \sin^2\aleph$. Squaring, $4\cos^2\aleph + 4\cos^4\aleph + \cos^6\aleph = \sin^4\aleph$. Substituting $\cos^2\aleph = 1 - \sin^2\aleph$ yields $-\sin^6\aleph + 7\sin^4\aleph - 15\sin^2\aleph + 9 = \sin^4\aleph$, and separating yields $\sin^6\aleph = 6\sin^4\aleph - 15\sin^2\aleph + 9$. $a + b + c = 0$.

14) The slopes of these lines can be represented by the vectors $\langle 1,2\rangle$ and $\langle 1,-7\rangle$. The dot product gives $\langle 1,2\rangle\cdot\langle 1,-7\rangle = |\langle 1,2\rangle||\langle 1,-7\rangle|\cos\theta_{bet}$. $1 - 14 = \sqrt{5}\sqrt{50}\cos\theta_{bet}$, so $\cos\theta_{bet} = -\frac{13}{5\sqrt{10}}$. This represents an obtuse angle; the cosine of the acute angle is $\frac{13}{5\sqrt{10}}$ and $\sin\theta_{ac} = \frac{9\sqrt{10}}{50}$.

15) Expanding out $\tan(\arctan z_1 + \arctan z_2 + \arctan z_3 + \arctan z_4)$ as $\tan\big((\arctan z_1 + \arctan z_2) + (\arctan z_3 + \arctan z_4)\big)$ with the tangent addition formula results in the fraction $\frac{\frac{z_1+z_2}{1-z_1z_2}+\frac{z_3+z_4}{1-z_3z_4}}{1-\frac{z_1+z_2}{1-z_1z_2}\cdot\frac{z_3+z_4}{1-z_3z_4}}$. FOILing and simplifying results in $\frac{z_1+z_2+z_3+z_4-z_1z_2z_3-z_1z_2z_4-z_1z_3z_4-z_2z_3z_4}{1-z_1z_2-z_1z_3-z_1z_4-z_2z_3-z_2z_4-z_3z_4+z_1z_2z_3z_4}$, which by Vieta's is equal to $\frac{e_1-e_3}{1-e_2+e_4}$. Using the polynomial, this is $\frac{-2+6}{1-2021+2022} = 2$.

16) The primitive period of $\sin\frac{x}{a}\cos\frac{x}{b}$ is $2\pi\,\mathrm{LCM}(a,b)$. The primitive period of each term is 40π and 112π respectively. The primitive period of the sum of two untranslated sinusoidal functions with periods a and b is $\mathrm{LCM}(a,b)$. Here, this is $\mathrm{LCM}(40\pi, 112\pi) = 560\pi$.

17) Note that $\tan\left(\frac{\pi}{4}-\theta\right) = \frac{1-\tan\theta}{1+\tan\theta} = -1 + \frac{2}{1+\tan\theta}$, so $\tan\theta = \frac{1-\tan\left(\frac{\pi}{4}-\theta\right)}{1+\tan\left(\frac{\pi}{4}-\theta\right)}$ and $\tan\theta + \tan\left(\frac{\pi}{4}-\theta\right) + \tan\theta\tan\left(\frac{\pi}{4}-\theta\right) + 1 = (1+\tan\theta)\left(1+\tan\left(\frac{\pi}{4}-\theta\right)\right) = 2$. The product is equal to $\prod_{n=0}^{45}(1+\tan n^\circ)\,(1+\tan(90^\circ-\theta)) = 2^{23}$, so $A + B = 25$.

18) A rose curve with an argument of θ being multiplied by an even integer n has $2n$ petals. $2\cdot 2022 = 4044$.

19) The reference angles of $\frac{1337\pi}{3}$ and $\frac{1337\pi}{6}$ are $\frac{5\pi}{3}$ and $\frac{5\pi}{6}$ respectively. In Cartesian, these points are $\left(\frac{3}{2}, -\frac{3\sqrt{3}}{2}\right)$ and $\left(-4\sqrt{3}, 4\right)$. Using the Distance Formula on these points gives a distance of $\sqrt{73 + 24\sqrt{3}}$. $A + H + S = 100$.

20) Converting to standard form, $r = \frac{3/4}{1+2\cos\theta}$. Plugging in $\theta = 0$ yields the point $\left(\frac{1}{4}, 0\right)$, and plugging in $\theta = \pi$ yields the point $\left(\frac{3}{4}, 0\right)$. The center of the conic is therefore at $\left(\frac{1}{2}, 0\right)$. One focus is at the origin, and the other focus is an equal distance away from the center at $(1,0)$. The other latus rectum passes through this point as part of the line $x = 1$.

21) The magnitude of $\sqrt{3} - 3i$ is $\sqrt{12}$, so $\sqrt{3} - 3i = \sqrt{12}\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) = \sqrt{12}\operatorname{cis}\left(-\frac{\pi}{3}\right)$. By De Moivre's, this taken to the power of 413 is $12^{413/2}\operatorname{cis}\left(-\frac{413\pi}{3}\right) = 12^{206}\sqrt{12}\operatorname{cis}\left(\frac{\pi}{3}\right) = 12^{206}\left(\sqrt{3} + 3i\right)$.

22) Using the complex definition of sine, $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$, we are solving $\frac{e^{iz} - e^{-iz}}{2i} = i$, or $e^{iz} - e^{-iz} + 2 = 0$. Multiplying by e^{iz} gives a quadratic in e^{iz}, $e^{2iz} - 2e^{iz} - 1 = 0$. Substituting $u = e^{iz}$ and solving for u, if $u^2 - 2u - 1 = 0$ then $u = 1 \pm \sqrt{2}$. Testing $u = 1 + \sqrt{2}$ gives $z = i\ln\left(\sqrt{2} - 1\right)$, but this is negative and not in the principal value range of the complex arcsine. Testing $u = 1 - \sqrt{2}$ gives $z = -i\ln\left(1 - \sqrt{2}\right) = i\ln\left(1 + \sqrt{2}\right)$, which is in the principal value range.

23) $\sin\theta + \cos\theta = \sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right)$. A polar function in this form is a parabola only if the constant term equals the coefficient of the sine or cosine term (or its negative), so $a = \sqrt{2}$.

24) $a_n + ib_n = a_{n+1}a_{n-1} + i^2 b_{n+1}b_{n-1} + ia_{n+1}b_{n-1} + ia_{n-1}b_{n+1} = (a_{n+1} + i_{n+1})(a_{n-1} + ib_{n+1})$. Let $S_n = a_n + ib_n$. Then $S_n = S_{n+1}S_{n-1}$. Similarly, $S_{n+1} = S_n S_{n+2} = S_{n+1}S_{n-1}S_{n+2}$, so $S_{n-1}S_{n+2} = 1$ and $S_{n-1} = \frac{1}{S_{n+2}}$. Index-shifting, $S_n = \frac{1}{S_{n+3}} = S_{n+6}$, so the sequence is periodic. $S_{2022} = S_0$, so $a_{2022} = 2$, $b_{2022} = 7$, and $a_{2022}^2 + b_{2022}^2 = 4 + 49 = 53$.

25) $(2+i)(3+i) = 6 + i^2 + 2i + 3i = 5 + 5i = 5\sqrt{2}\operatorname{cis}\frac{\pi}{4}$. This equals $\left(5\sqrt{2}\operatorname{cis}\frac{\pi}{4}\right)^4 (3+i)^2 = (2500\operatorname{cis}\pi)(9 + 6i + i^2) = -20000 - 15000i$. The sum of the digits of $A + B = 35000$ is 8.

26) $r\sin(\theta - \arctan 2) = \frac{1}{\sqrt{5}}$, so by the sine addition formula, $r(\sin\theta\cos(\arctan 2) - \cos\theta\sin(\arctan 2)) = \frac{1}{\sqrt{5}}$. $\sin(\arctan 2) = \frac{2}{\sqrt{5}}$ and $\cos(\arctan 2) = \frac{1}{\sqrt{5}}$, so this simplifies to $r\cos\theta - 2r\sin\theta = 1$. Converting from polar, $f(x) - 2x = 1$, or $f(x) = 2x + 1$. $f(2) = 5$.

27) $\ln(-\sqrt{3} - i) = \ln(2) + \ln\left(-\frac{\sqrt{3}}{2} - \frac{i}{2}\right) = \ln(2) + \ln e^{-5i\pi/6} = \ln(2) - \frac{5i\pi}{6}$. $A + B + C = 13$. Note that the imaginary part of the principal value of $\ln(z)$ always lies in the range $(-\pi, \pi]$.

28) This is an ellipse with foci at $(0, -i)$ and $(0, i)$ with major axis length 4. The semimajor axis has length 2, and the focal radius is 1. Solving $1^2 = 2^2 - x^2$ gives a semiminor axis of length $\sqrt{3}$. Thus, the area of the ellipse is $2\pi\sqrt{3}$.

29) The determinant is $\operatorname{cis}\frac{\pi}{4} \cdot \operatorname{cis}\frac{5\pi}{4} \cdot \operatorname{cis}\frac{\pi}{2} = \operatorname{cis} 2\pi = 1$. This is an easy triangular matrix to invert.

$$\begin{bmatrix} \operatorname{cis}\pi/4 & \operatorname{cis}3\pi/2 & \operatorname{cis}\pi/2 \\ 0 & \operatorname{cis}5\pi/4 & \operatorname{cis}\pi/4 \\ 0 & 0 & \operatorname{cis}\pi/2 \end{bmatrix} \lll\ggg \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \lll\ggg R_1 \div= \operatorname{cis}\pi/4$$

$$\begin{bmatrix} 1 & \operatorname{cis}5\pi/4 & \operatorname{cis}\pi/4 \\ 0 & \operatorname{cis}5\pi/4 & \operatorname{cis}\pi/4 \\ 0 & 0 & \operatorname{cis}\pi/2 \end{bmatrix} \lll\ggg \begin{bmatrix} \operatorname{cis}-\pi/4 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \lll\ggg R_1 -= R_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \operatorname{cis}5\pi/4 & \operatorname{cis}\pi/4 \\ 0 & 0 & \operatorname{cis}\pi/2 \end{bmatrix} \lll\ggg \begin{bmatrix} \operatorname{cis}-\pi/4 & -1 & 0 \\ 0 & \operatorname{cis}3\pi/4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \lll\ggg R_2 \times= \operatorname{cis}3\pi/4$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \operatorname{cis}\pi \\ 0 & 0 & \operatorname{cis}\pi/2 \end{bmatrix} \lll\ggg \begin{bmatrix} \operatorname{cis}-\pi/4 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \lll\ggg \text{Convert left matrix to rectangular}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & i \end{bmatrix} \lll\ggg \begin{bmatrix} \operatorname{cis}-\pi/4 & -1 & 0 \\ 0 & \operatorname{cis}3\pi/4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \lll\ggg R_2 -= iR_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{bmatrix} \lll\ggg \begin{bmatrix} \operatorname{cis}-\pi/4 & -1 & 0 \\ 0 & \operatorname{cis}3\pi/4 & -i \\ 0 & 0 & 1 \end{bmatrix} \lll\ggg R_3 \div= i$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \lll\ggg \begin{bmatrix} \operatorname{cis}-\pi/4 & -1 & 0 \\ 0 & \operatorname{cis}3\pi/4 & -i \\ 0 & 0 & -i \end{bmatrix} \lll\ggg \text{Convert right matrix to rectangular}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \lll\ggg \begin{bmatrix} 1/\sqrt{2} - i/\sqrt{2} & -1 & 0 \\ 0 & -1/\sqrt{2} + i/\sqrt{2} & -i \\ 0 & 0 & -i \end{bmatrix}$$

The sum of the entries in the inverse matrix is $-1-2i$. $A+B=3$.

30) The areas of triangle OPR, sector POQ, and triangle OPS are $\frac{\sin\theta}{2}$, $\frac{\theta}{2}$, and $\frac{\tan\theta}{2}$ respectively. For all values in the range $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ except at 0, these are ordered, so $\frac{\sin\theta}{2} < \frac{\theta}{2} < \frac{\tan\theta}{2}$. Multiplying by 2 and dividing by $\sin\theta$ (note that θ is always positive for area), this becomes $1 < \frac{\theta}{\sin\theta} < \sec\theta$. Inverting, $1 > \frac{\sin\theta}{\theta} > \cos\theta$. As θ becomes close to 0, $\cos\theta$ approaches 1, and $\frac{\sin\theta}{\theta}$ becomes sandwiched between a value approaching 1 and 1. Thus, $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$. $\left|\frac{\sqrt{1-\cos 2\theta}}{4\theta}\right| = \frac{\sqrt{2}\sin\theta}{4\theta}$, so $\lim_{\theta\to 0}\left|\frac{\sqrt{1-\cos 2\theta}}{4\theta}\right| = \lim_{\theta\to 0}\frac{\sqrt{2}\sin\theta}{4\theta} = \frac{\sqrt{2}}{4}\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = \frac{\sqrt{2}}{4}$.