Alpha

Trigonometry

Test #121

Directions:

1. Fill out the top section of the Round 1 Google Form answer sheet and select **Alpha- Trigonometry** as the test. Do not abbreviate your school name. Enter an email address that will accept outside emails (some school email addresses do not).

2. Scoring for this test is 5 times the number correct plus the number omitted.

3. TURN OFF ALL CELL PHONES.

4. No calculators may be used on this test.

5. Any inappropriate behavior or any form of cheating will lead to a ban of the student and/or school from future National Conventions, disqualification of the student and/or school from this Convention, at the discretion of the Mu Alpha Theta Governing Council.

6. If a student believes a test item is defective, select “E) NOTA” and file a dispute explaining why.

7. If an answer choice is incomplete, it is considered incorrect. For example, if an equation has three solutions, an answer choice containing only two of those solutions is incorrect.

8. If a problem has wording like “which of the following could be” or “what is one solution of”, an answer choice providing one of the possibilities is considered to be correct. Do not select “E) NOTA” in that instance.

9. If a problem has multiple equivalent answers, any of those answers will be counted as correct, even if one answer choice is in a simpler format than another. Do not select “E) NOTA” in that instance.

10. Unless a question asks for an approximation or a rounded answer, give the exact answer.

All uppercase letter variables are positive integers unless otherwise stated. All fractions containing uppercase letter variables are in lowest terms. NOTA means "None of the Above." All functions should be interpreted as having complex domains and codomains.

~~~~~~~~ *Good luck, and have fun!* ~~~~~~~~

1. Which of the following ordered pairs represents the primitive period and then the amplitude of the function $f(x) = 5\cos 6x - 12\sin 6x$?

A. $\left(\frac{\pi}{6}, 12\right)$
B. $\left(\frac{\pi}{6}, 13\right)$
C. $\left(\frac{\pi}{3}, 12\right)$
D. $\left(\frac{\pi}{3}, 13\right)$
E. NOTA

2. If $\cos\theta = \frac{12}{13}$ and $\tan\phi = -\frac{4}{3}$ for $0 \le \theta, \phi \le \pi$, find $\sec(\phi - \theta)$.

A. $-\frac{65}{16}$
B. $-\frac{65}{33}$
C. $-\frac{65}{56}$
D. $-\frac{65}{63}$
E. NOTA

3. Determine the primitive period of the function $f(x) = \cos(\sin\theta)$.

A. $\pi/2$
B. π
C. $3\pi/2$
D. 2π
E. NOTA

4. Find the number of solutions to $2^{\sin\theta - \cos\theta} = \tan\theta$ in the interval $[0, 2\pi)$.

A. 2
B. 3
C. 4
D. 6
E. NOTA

5. If the sine of the smallest angle in a triangle with side lengths 4, 5, and 7 is equal to $\frac{A\sqrt{B}}{C}$ for squarefree B, find $A + B + C$.

A. 27
B. 39
C. 43
D. 49
E. NOTA
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6. In a triangle with side lengths a, b, and c, $\frac{a+b}{21} = \frac{b+c}{22} = \frac{c+a}{23}$. Find the cosine of the smallest angle of the triangle.

A. $\frac{7}{20}$
B. $\frac{41}{80}$
C. $\frac{23}{40}$
D. $\frac{5}{8}$
E. NOTA

7. Find the sum of all positive integers a such that the value of x that satisfies the equation $\arctan a - \arctan x = \operatorname{arccot} 7$ is also a positive integer.

A. 21
B. 42
C. 64
D. 77
E. NOTA

8. Find the volume of a tetrahedron with vertices at $(1,2,3)$, $(4,2,0)$, $(3,5,-2)$, and $(-1,6,9)$.

A. 9
B. $6\sqrt{3}$
C. 12
D. $8\sqrt{3}$
E. NOTA

9. Find the sum of the solutions to $\cos x\,(1 - 4\sin^2 x) = \frac{1}{\sqrt{3}}$ in the interval $[0,2\pi)$.

A. 3π
B. $\frac{13\pi}{3}$
C. $\frac{9\pi}{2}$
D. 6π
E. NOTA

10. If $\theta_1 + \theta_2 = \frac{5\pi}{4}$, which of the following are always equal to 1?

I: $\frac{\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2}{\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2}$

II: $\tan\theta_1 + \tan\theta_2 + \tan\theta_1\tan\theta_2$, where $\tan\theta_1$ and $\tan\theta_2$ are defined

III: $\frac{\cos(\theta_1-\theta_2)+\sin(\theta_1+\theta_2)}{\cos(\theta_1+\theta_2)+\cos(\theta_1-\theta_2)}$

A. I ONLY
B. I and III ONLY
C. II ONLY
D. I, II, AND III
E. NOTA

11. If $60^{\sin^2\theta}$ and $60^{\cos^2\theta}$ are both positive integers, find the smallest possible value of their sum.

A. 12
B. 16
C. 17
D. 20
E. NOTA

12. The values $\cot(\alpha-\beta)$, $3\cot(\alpha)$, and $\cot(\alpha+\beta)$ form an arithmetic progression in that order. Find $\sin^2(\alpha)\csc^2(\beta)$, given that α and β are not integer multiples of $\frac{\pi}{2}$.

A. $\frac{1}{2}$
B. 1
C. $\frac{3}{2}$
D. 2
E. NOTA

13. For some acute angle $\aleph$, $2\cos\aleph+\cos^2\aleph+\cos^3\aleph=1$. There exists a unique set of integers a, b, and c such that $\sin^6\aleph=a\sin^4\aleph+b\sin^2\aleph+c$. Find $a+b+c$.

A. 0
B. 2
C. 1
D. 3
E. NOTA

14. Find the sine of the acute angle between the lines $y=2x+\sqrt{2}$ and $y=-7x-9$.

A. $\frac{2\sqrt{5}}{25}$
B. $\frac{\sqrt{2}}{5}$
C. $\frac{9\sqrt{10}}{50}$
D. $\frac{13\sqrt{10}}{50}$
E. NOTA

15. Let the complex roots of the polynomial $f(z)=z^4+2z^3+2021z^2+6z+2022=0$ be z_1, z_2, z_3, and z_4. Evaluate $\tan\left(\sum_{n=1}^{4}\arctan z_n\right)$.

A. 2
B. 5
C. 6
D. 9
E. NOTA

16. Determine the primitive period of the function $f(x)=\sin\frac{x}{4}\cos\frac{x}{5}+\sin\frac{x}{8}\cos\frac{x}{7}$.

A. 172π
B. 560π
C. 1120π
D. 2240π
E. NOTA

17. If $\prod_{n=0}^{45}(1+\tan n^\circ) = A^B$ for prime A, find $A+B$.

A. 18
B. 21
C. 25
D. 27
E. NOTA

18. Find the number of petals in the rose curve given by $r = 2\sin 2022\theta$.

A. 2
B. 1011
C. 2022
D. 4044
E. NOTA

19. If the Cartesian distance between the polar coordinates $\left(3, \frac{1337\pi}{3}\right)$ and $\left(8, \frac{1337\pi}{6}\right)$ is equal to $\sqrt{A + H\sqrt{S}}$ where S is square-free, find $A + H + S$.

A. 52
B. 100
C. 121
D. 151
E. NOTA

20. One latus rectum of the hyperbola given by the equation $r = \frac{3}{4+8\cos\theta}$ passes through the origin. The other one lies on the line $x = k$. Find k.

A. $\frac{3}{4}$
B. 1
C. $\frac{5}{4}$
D. $\frac{4}{3}$
E. NOTA

21. Evaluate: $\left(\sqrt{3} - 3i\right)^{413}$.

A. $12^{206}\left(3 - i\sqrt{3}\right)$
B. $12^{206}\left(3 + i\sqrt{3}\right)$
C. $12^{206}\left(\sqrt{3} - 3i\right)$
D. $12^{206}\left(\sqrt{3} + 3i\right)$
E. NOTA

22. Which of the following is a solution to the equation $\sin z = i$?

A. $i \ln\sqrt{2}$
B. $i \ln\left(\frac{1}{2} + \sqrt{2}\right)$
C. $i \ln(2)$
D. $i \ln\left(1 + \sqrt{2}\right)$
E. NOTA

23. Consider the polar function $r(a, \theta) = \frac{2}{a+\sin\theta+\cos\theta}$. Find the greatest value of a such that this represents a parabola.

A. 0
B. 1
C. $\sqrt{2}$
D. 2
E. NOTA

24. Two sequences $\{a_n\}$ and $\{b_n\}$ are given by the following recursive relation for all $n \geq 1$.

$$a_n = a_{n+1}a_{n-1} - b_{n+1}b_{n-1}$$
$$b_n = a_{n+1}b_{n-1} + a_{n-1}b_{n+1}$$

$a_0 = 2$, $b_0 = 7$, $a_1 = 1$, and $b_1 = 8$. Find $a_{2022}^2 + b_{2022}^2$.

Hint: Consider complex numbers of the form $a + bi$.

A. 53
B. 65
C. 79
D. 610
E. NOTA

25. If $(2+i)^4(3+i)^6 = -(A + Bi)$, find the sum of the digits of $A + B$.

A. 1
B. 4
C. 7
D. 8
E. NOTA

26. The graph of $r = \frac{1}{\sqrt{5}}\csc(\theta - \arctan 2)$ in the Cartesian plane is a function $f(x)$. Find $f(2)$.

A. 5
B. 6
C. 7
D. 8
E. NOTA

27. Given that the principal argument of a complex number always lies in $(-\pi, \pi]$, if $\ln(-\sqrt{3} - i)$ can be expressed as $\ln A - \frac{Bi\pi}{C}$. Find $A + B + C$.

A. 9
B. 11
C. 13
D. 15
E. NOTA

28. Find the area of the ellipse given by $|z+i|+|z-i|=4$ in the Argand plane.

A. $2\pi\sqrt{3}$
B. 4π
C. 6π
D. $4\pi\sqrt{3}$
E. NOTA

29. If the sum of the entries in $\begin{bmatrix} \operatorname{cis}\frac{\pi}{4} & \operatorname{cis}\frac{3\pi}{2} & \operatorname{cis}\frac{\pi}{2} \\ 0 & \operatorname{cis}\frac{5\pi}{4} & \operatorname{cis}\frac{\pi}{4} \\ 0 & 0 & \operatorname{cis}\frac{\pi}{2} \end{bmatrix}^{-1}$ is $-(A+Bi)$, find $A+B$.

A. 2
B. 3
C. 4
D. 5
E. NOTA

30. In the diagram below, $\overline{OP}$ is a radius of the unit circle with center O. $\overline{PR}$ is a vertical line segment with R on the x-axis. Q is the point $(1,0)$. S is the intersection of the line tangent to the unit circle at P with the x-axis. Consider the areas of triangle OPR, sector POQ, and triangle OPS. Noting that for all $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$, these are written in increasing order (except at $\theta=0$), find $\lim_{\theta\to 0}\left|\frac{\sqrt{1-\cos 2\theta}}{4\theta}\right|$.

Note: The notation $\lim_{\theta\to 0}$ *is to be interpreted as the value of the attached expression as* θ *becomes infinitesimally small in magnitude, approaching* 0 *from both the left and the right.*

A. 0
B. $\frac{1}{2}$
C. $\frac{1}{4}$
D. $\frac{\sqrt{2}}{2}$
E. NOTA