**2012 – 2013 Log1 Contest Round 2**
**Theta Sequences & Series**

Name: ________________

| 4 points each | | |
|---|---|---|
| 1 | If $a_n = 3n + 2^n$, find the value of $a_8$. | |
| 2 | Define a "Fibonacci-like" sequence as any sequence such that any term in the sequence after the second term is the sum of its immediate two preceding terms. Find the 10th term in a Fibonacci-like sequence whose first term is 5 and whose second term is 12. | |
| 3 | How many of the first 100 rows of Pascal's Triangle contain an odd number of entries that are odd integers? Count the row with a single 1 as the first row. | |
| 4 | In an arithmetic sequence, the first term is 2 and the common difference is 3. Find the 2013th term in this sequence. | |
| 5 | Find the sum of the infinite geometric series: $3 + 1 + \frac{1}{3} + \frac{1}{9} + \ldots$ | |

| 5 points each | | |
|---|---|---|
| 6 | Three numbers whose sum is 30 form an increasing arithmetic sequence. The product of the greatest two numbers is three times the product of the least two numbers. What is the least of the three numbers? | |
| 7 | In a geometric sequence of positive numbers, the eighth term is 17 and the 12th term is $\frac{10625}{16}$. Find the common ratio of the sequence. | |
| 8 | A ball is dropped off a 90-foot tall building. Each time the ball falls, it only bounces off the ground; the ball also only moves in a vertical direction. Additionally, the ball rebounds to one-third of the height from which it fell immediately before bouncing. Find the total distance, in feet, the ball travels before coming to rest. | |
| 9 | Three numbers have a product of 1728. When arranged in non-decreasing order, the three numbers form an arithmetic and geometric sequence. List the three numbers. | |
| 10 | The roots of $y = x^3 + 3x^2 - bx - 8$ form an arithmetic sequence. Find the value of $b$. | |

| 6 points each | | |
|---|---|---|
| 11 | Find the value of $\sum_{n=1}^{\infty} \frac{4n}{3^n}$. | |
| 12 | The sum of three distinct numbers $x$, $y$, and $z$ is 24. The sequence $x, y, z$ is geometric while the sequence $y, x, z$ is arithmetic. Find the value of $y$. | |
| 13 | If $a_{n-1} = \binom{n}{2}$, where $n \geq 2$ is an integer, find the greatest value of $\frac{a_{n+1}}{a_n}$. | |
| 14 | In a sequence $\langle a_n \rangle$, where $n$ is a positive integer, $a_{n+1} =$ the sum of the squares of the digits of $a_n$. If $a_1 = 2013$, find the first number in the sequence that occurs twice. | |
| 15 | From the set $\{1,2,3,4,5\}$, five digits are chosen in order, with replacement, to form a sequence. How many of these five-digit sequences contain at least one repeated digit? | |

# 2012 – 2013 Log1 Contest Round 2
## Alpha Sequences & Series

Name: ________________

| 4 points each | | |
|---|---|---|
| 1 | If $a_n = 3n + 2^n$, find the value of $a_8$. | |
| 2 | Define a "Fibonacci-like" sequence as any sequence such that any term in the sequence after the second term is the sum of its immediate two preceding terms. Find the 10th term in a Fibonacci-like sequence whose first term is 5 and whose second term is 12. | |
| 3 | How many of the first 100 rows of Pascal's Triangle contain an odd number of entries that are odd integers? Count the row with a single 1 as the first row. | |
| 4 | Find the sum of the infinite geometric series: $3+1+\frac{1}{3}+\frac{1}{9}+...$ | |
| 5 | Three numbers whose sum is 30 form an increasing arithmetic sequence. The product of the greatest two numbers is three times the product of the least two numbers. What is the least of the three numbers? | |

| 5 points each | | |
|---|---|---|
| 6 | In a geometric sequence of positive numbers, the eighth term is 17 and the 12th term is $\frac{10625}{16}$. Find the common ratio of the sequence. | |
| 7 | A ball is dropped off a 90-foot tall building. Each time the ball falls, it only bounces off the ground; the ball also only moves in a vertical direction. Additionally, the ball rebounds to one-third of the height from which it fell immediately before bouncing. Find the total distance, in feet, the ball travels before coming to rest. | |
| 8 | Three numbers have a product of 1728. When arranged in non-decreasing order, the three numbers form an arithmetic and geometric sequence. List the three numbers. | |
| 9 | The roots of $y = x^3 + 3x^2 - bx - 8$ form an arithmetic sequence. Find the value of $b$. | |
| 10 | Find the value of $\sum_{n=1}^{\infty} \frac{4n}{3^n}$. | |

| 6 points each | | |
|---|---|---|
| 11 | Find the value of $\sum_{n=1}^{\infty}\frac{F_n}{3^n}$, where $F_n$ is the $n$th Fibonacci number with $F_1 = F_2 = 1$. | |
| 12 | The sum of three distinct numbers $x$, $y$, and $z$ is 24. The sequence $x, y, z$ is geometric while the sequence $y, x, z$ is arithmetic. Find the value of $y$. | |
| 13 | Find the smallest positive value of $\theta$ such that $\sum_{n=0}^{\infty}\left(\cos\theta \cdot \sin^n \theta\right) = \sqrt{3}$. | |
| 14 | In a sequence $\langle a_n \rangle$, where $n$ is a positive integer, $a_{n+1} =$ the sum of the squares of the digits of $a_n$. If $a_1 = 2013$, find the first number in the sequence that occurs twice. | |
| 15 | From the set $\{1,2,3,4,5\}$, five digits are chosen in order, with replacement, to form a sequence. How many of these five-digit sequences contain at least one repeated digit? | |

**2012 – 2013 Log1 Contest Round 2**
**Mu Sequences & Series**

Name: ________________

| | 4 points each | |
|---|---|---|
| 1 | If $a_n = 3n + 2^n$, find the value of $a_8$. | |
| 2 | Define a "Fibonacci-like" sequence as any sequence such that any term in the sequence after the second term is the sum of its immediate two preceding terms. Find the 10th term in a Fibonacci-like sequence whose first term is 5 and whose second term is 12. | |
| 3 | How many of the first 100 rows of Pascal's Triangle contain an odd number of entries that are odd integers? Count the row with a single 1 as the first row. | |
| 4 | Find the sum of the infinite geometric series: $3 + 1 + \frac{1}{3} + \frac{1}{9} + \ldots$ | |
| 5 | Three numbers whose sum is 30 form an increasing arithmetic sequence. The product of the greatest two numbers is three times the product of the least two numbers. What is the least of the three numbers? | |

| | 5 points each | |
|---|---|---|
| 6 | Let the $n$th term of a sequence be defined by $a_n = \int_0^{n\pi/2} (\sin x)\,dx$. Find the value of $\sum_{n=1}^{2013} a_n$. | |
| 7 | A ball is dropped off a 90-foot tall building. Each time the ball falls, it only bounces off the ground; the ball also only moves in a vertical direction. Additionally, the ball rebounds to one-third of the height from which it fell immediately before bouncing. Find the total distance, in feet, the ball travels before coming to rest. | |
| 8 | Three numbers have a product of 1728. When arranged in non-decreasing order, the three numbers form an arithmetic and geometric sequence. List the three numbers. | |
| 9 | Find the value of $\sum_{n=1}^{\infty} \frac{4n}{3^n}$. | |
| 10 | Find the value of $\sum_{n=1}^{\infty} \frac{F_n}{3^n}$, where $F_n$ is the $n$th Fibonacci number with $F_1 = F_2 = 1$. | |

| 6 points each | | |
|---|---|---|
| 11 | The sum of three distinct numbers $x$, $y$, and $z$ is 24. The sequence $x$, $y$, $z$ is geometric while the sequence $y$, $x$, $z$ is arithmetic. Find the value of $y$. | |
| 12 | Find the smallest positive value of $\theta$ such that $\sum_{n=0}^{\infty}\left(\cos\theta\cdot\sin^{n}\theta\right)=\sqrt{3}$. | |
| 13 | In a sequence $\langle a_n \rangle$, where $n$ is a positive integer, $a_{n+1}=$ the sum of the squares of the digits of $a_n$. If $a_1=2013$, find the first number in the sequence that occurs twice. | |
| 14 | From the set $\{1,2,3,4,5\}$, five digits are chosen in order, with replacement, to form a sequence. How many of these five-digit sequences contain at least one repeated digit? | |
| 15 | Given that $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$, find the value of $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$. | |

**2012 – 2013 Log1 Contest Round 2**
**Theta Sequences & Series**

Name: ________________

| 4 points each | | |
|---|---|---|
| 1 | If $a_n = 3n + 2^n$, find the value of $a_8$. | 280 |
| 2 | Define a "Fibonacci-like" sequence as any sequence such that any term in the sequence after the second term is the sum of its immediate two preceding terms. Find the 10th term in a Fibonacci-like sequence whose first term is 5 and whose second term is 12. | 513 |
| 3 | How many of the first 100 rows of Pascal's Triangle contain an odd number of entries that are odd integers? Count the row with a single 1 as the first row. | 1 |
| 4 | In an arithmetic sequence, the first term is 2 and the common difference is 3. Find the 2013th term in this sequence. | 6038 |
| 5 | Find the sum of the infinite geometric series: $3+1+\frac{1}{3}+\frac{1}{9}+...$ | 4.5 or $\frac{9}{2}$ |

| 5 points each | | |
|---|---|---|
| 6 | Three numbers whose sum is 30 form an increasing arithmetic sequence. The product of the greatest two numbers is three times the product of the least two numbers. What is the least of the three numbers? | 5 |
| 7 | In a geometric sequence of positive numbers, the eighth term is 17 and the 12th term is $\frac{10625}{16}$. Find the common ratio of the sequence. | 2.5 or $\frac{5}{2}$ |
| 8 | A ball is dropped off a 90-foot tall building. Each time the ball falls, it only bounces off the ground; the ball also only moves in a vertical direction. Additionally, the ball rebounds to one-third of the height from which it fell immediately before bouncing. Find the total distance, in feet, the ball travels before coming to rest. | 180 |
| 9 | Three numbers have a product of 1728. When arranged in non-decreasing order, the three numbers form an arithmetic and geometric sequence. List the three numbers. | 12, 12, 12 |
| 10 | The roots of $y = x^3 + 3x^2 - bx - 8$ form an arithmetic sequence. Find the value of $b$. | 6 |

| 6 points each | | |
|---|---|---|
| 11 | Find the value of $\sum_{n=1}^{\infty} \frac{4n}{3^n}$. | 3 |
| 12 | The sum of three distinct numbers $x$, $y$, and $z$ is 24. The sequence $x$, $y$, $z$ is geometric while the sequence $y$, $x$, $z$ is arithmetic. Find the value of $y$. | –16 |
| 13 | If $a_{n-1} = \binom{n}{2}$, where $n \geq 2$ is an integer, find the greatest value of $\frac{a_{n+1}}{a_n}$. | 2 |
| 14 | In a sequence $\langle a_n \rangle$, where $n$ is a positive integer, $a_{n+1} =$ the sum of the squares of the digits of $a_n$. If $a_1 = 2013$, find the first number in the sequence that occurs twice. | 89 |
| 15 | From the set $\{1,2,3,4,5\}$, five digits are chosen in order, with replacement, to form a sequence. How many of these five-digit sequences contain at least one repeated digit? | 3005 |

**2012 – 2013 Log1 Contest Round 2**
**Alpha Sequences & Series**

Name: ________________

| | 4 points each | |
|---|---|---|
| 1 | If $a_n = 3n + 2^n$, find the value of $a_8$. | 280 |
| 2 | Define a "Fibonacci-like" sequence as any sequence such that any term in the sequence after the second term is the sum of its immediate two preceding terms. Find the 10th term in a Fibonacci-like sequence whose first term is 5 and whose second term is 12. | 513 |
| 3 | How many of the first 100 rows of Pascal's Triangle contain an odd number of entries that are odd integers? Count the row with a single 1 as the first row. | 1 |
| 4 | Find the sum of the infinite geometric series: $3+1+\frac{1}{3}+\frac{1}{9}+\ldots$ | 4.5 or $\frac{9}{2}$ |
| 5 | Three numbers whose sum is 30 form an increasing arithmetic sequence. The product of the greatest two numbers is three times the product of the least two numbers. What is the least of the three numbers? | 5 |

| | 5 points each | |
|---|---|---|
| 6 | In a geometric sequence of positive numbers, the eighth term is 17 and the 12th term is $\frac{10625}{16}$. Find the common ratio of the sequence. | 2.5 or $\frac{5}{2}$ |
| 7 | A ball is dropped off a 90-foot tall building. Each time the ball falls, it only bounces off the ground; the ball also only moves in a vertical direction. Additionally, the ball rebounds to one-third of the height from which it fell immediately before bouncing. Find the total distance, in feet, the ball travels before coming to rest. | 180 |
| 8 | Three numbers have a product of 1728. When arranged in non-decreasing order, the three numbers form an arithmetic and geometric sequence. List the three numbers. | 12, 12, 12 |
| 9 | The roots of $y = x^3 + 3x^2 - bx - 8$ form an arithmetic sequence. Find the value of $b$. | 6 |
| 10 | Find the value of $\sum_{n=1}^{\infty} \frac{4n}{3^n}$. | 3 |

| 6 points each | | |
|---|---|---|
| 11 | Find the value of $\sum_{n=1}^{\infty} \frac{F_n}{3^n}$, where $F_n$ is the $n$th Fibonacci number with $F_1 = F_2 = 1$. | 0.6 or $\frac{3}{5}$ |
| 12 | The sum of three distinct numbers $x$, $y$, and $z$ is 24. The sequence $x, y, z$ is geometric while the sequence $y, x, z$ is arithmetic. Find the value of $y$. | $-16$ |
| 13 | Find the smallest positive value of $\theta$ such that $\sum_{n=0}^{\infty} \left( \cos\theta \cdot \sin^n \theta \right) = \sqrt{3}$. | $\frac{\pi}{6}$ |
| 14 | In a sequence $\langle a_n \rangle$, where $n$ is a positive integer, $a_{n+1} =$ the sum of the squares of the digits of $a_n$. If $a_1 = 2013$, find the first number in the sequence that occurs twice. | 89 |
| 15 | From the set $\{1,2,3,4,5\}$, five digits are chosen in order, with replacement, to form a sequence. How many of these five-digit sequences contain at least one repeated digit? | 3005 |

**2012 – 2013 Log1 Contest Round 2**
**Mu Sequences & Series**

Name: ________________

| | 4 points each | |
|---|---|---|
| 1 | If $a_n = 3n + 2^n$, find the value of $a_8$. | 280 |
| 2 | Define a "Fibonacci-like" sequence as any sequence such that any term in the sequence after the second term is the sum of its immediate two preceding terms. Find the 10th term in a Fibonacci-like sequence whose first term is 5 and whose second term is 12. | 513 |
| 3 | How many of the first 100 rows of Pascal's Triangle contain an odd number of entries that are odd integers? Count the row with a single 1 as the first row. | 1 |
| 4 | Find the sum of the infinite geometric series: $3 + 1 + \frac{1}{3} + \frac{1}{9} + \ldots$ | 4.5 or $\frac{9}{2}$ |
| 5 | Three numbers whose sum is 30 form an increasing arithmetic sequence. The product of the greatest two numbers is three times the product of the least two numbers. What is the least of the three numbers? | 5 |

| | 5 points each | |
|---|---|---|
| 6 | Let the *n*th term of a sequence be defined by $a_n = \int_0^{n\pi/2} (\sin x)\,dx$. Find the value of $\sum_{n=1}^{2013} a_n$. | 2013 |
| 7 | A ball is dropped off a 90-foot tall building. Each time the ball falls, it only bounces off the ground; the ball also only moves in a vertical direction. Additionally, the ball rebounds to one-third of the height from which it fell immediately before bouncing. Find the total distance, in feet, the ball travels before coming to rest. | 180 |
| 8 | Three numbers have a product of 1728. When arranged in non-decreasing order, the three numbers form an arithmetic and geometric sequence. List the three numbers. | 12, 12, 12 |
| 9 | Find the value of $\sum_{n=1}^{\infty} \frac{4n}{3^n}$. | 3 |
| 10 | Find the value of $\sum_{n=1}^{\infty} \frac{F_n}{3^n}$, where $F_n$ is the *n*th Fibonacci number with $F_1 = F_2 = 1$. | 0.6 or $\frac{3}{5}$ |

| 6 points each | | |
|---|---|---|
| 11 | The sum of three distinct numbers $x, y,$ and $z$ is 24. The sequence $x, y, z$ is geometric while the sequence $y, x, z$ is arithmetic. Find the value of $y$. | $-16$ |
| 12 | Find the smallest positive value of $\theta$ such that $\sum_{n=0}^{\infty}\left(\cos\theta\cdot\sin^{n}\theta\right)=\sqrt{3}$. | $\frac{\pi}{6}$ |
| 13 | In a sequence $\langle a_n \rangle$, where $n$ is a positive integer, $a_{n+1}=$ the sum of the squares of the digits of $a_n$. If $a_1=2013$, find the first number in the sequence that occurs twice. | 89 |
| 14 | From the set $\{1,2,3,4,5\}$, five digits are chosen in order, with replacement, to form a sequence. How many of these five-digit sequences contain at least one repeated digit? | 3005 |
| 15 | Given that $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$, find the value of $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$. | $\frac{\pi^2}{12}$ |

**2012 – 2013 Log1 Contest Round 2**
**Sequences & Series Solutions**

| Mu | Al | Th | Solution |
|---|---|---|---|
| 1 | 1 | 1 | $a_8 = 3\cdot 8 + 2^8 = 24 + 256 = 280$ |
| 2 | 2 | 2 | The terms in this sequence would be 5, 12, 17, 29, 46, 75, 121, 196, 317, 513,…, so the 10th term is 513. |
| 3 | 3 | 3 | The first row obviously has an odd number of odd numbers since it contains only 1. For all other rows, since $\binom{n}{r} = \binom{n}{n-r}$, if an odd number appears once, it also appears twice. The only term that must be checked is a term of the form $\binom{2n}{n}$, since that term will only appear once in its row. However, since the sum of a row of Pascal's Triangle must be a power of 2 (even), and all other terms appear twice (which will have an even sum), $\binom{2n}{n}$ must also be even. Therefore, only the first row has an odd number of odd entries of all rows, not just the first 100 rows. |
| | | 4 | $a_{2013} = 2 + 3(2013-1) = 2 + 3(2012) = 2 + 6036 = 6038$ |
| 4 | 4 | 5 | The sum is $\frac{3}{1-1/3} = \frac{3}{2/3} = 3\cdot\frac{3}{2} = \frac{9}{2}$. |
| 5 | 5 | 6 | The three numbers can be written as $10-d,\ 10,\ 10+d$, where $d$ is the common ratio of the arithmetic sequence. Therefore, $10(10+d) = 3(10(10-d)) \Rightarrow 10+d = 30-3d$ $\Rightarrow 4d = 20 \Rightarrow d = 5$, so the least number is $10-5=5$. |
| | 6 | 7 | $r^4 = \frac{10625}{16\cdot 17} = \frac{625}{16} \Rightarrow r = \frac{5}{2}$ (since the sequence consisted of only positive terms) |
| 6 | | | $a_n = \int_0^{n\pi/2} (\sin x)\,dx = -\cos x\Big\vert_0^{n\pi/2} = -\cos\frac{n\pi}{2} + 1$, so the terms of the sequence are 1, 2, 1, 0, …, having those 4 numbers repeat. Since the sum of the four numbers is 4, the sum of the first 2012 numbers is 2012, then add 1 for the 2013th term to get 2013. |
| 7 | 7 | 8 | Since the series of downward and upward motions is geometric, the sum of the downward motion is $\frac{90}{1-1/3} = \frac{90}{2/3} = 90\cdot\frac{3}{2} = 135$, and the sum of the upward motion is $\frac{30}{1-1/3} = \frac{30}{2/3} = 30\cdot\frac{3}{2} = 45$, making the total distance traveled $135+45=180$ feet. |
| 8 | 8 | 9 | Let the term be $a-d,\ a,$ and $a+d$ because they are in arithmetic progression. Since they are also in geometric progression, $\frac{a}{a-d} = \frac{a+d}{a} \Rightarrow a^2 = a^2 - d^2 \Rightarrow d = 0$, so the three terms are all the same. Therefore, $a^3 = 1728 \Rightarrow a = 12$, and the three numbers are 12, 12, and 12. |
| | 9 | 10 | Since the sum of the roots is $-3$, $-1$ must be the middle root. Therefore, $0 = (-1)^3$ $+3(-1)^2 - b(-1) - 8 = -1 + 3 + b - 8 = b - 6 \Rightarrow b = 6$. |

| | | | |
|---|---|---|---|
| 9 | 10 | 11 | Let $S=\sum_{n=1}^{\infty}\frac{4n}{3^n}=\frac{4}{3}+\frac{8}{9}+\frac{12}{27}+\frac{16}{81}+\ldots$. Multiply this equation by $\frac{1}{3}$ on both sides to yield $\frac{1}{3}S=\frac{4}{9}+\frac{8}{27}+\frac{12}{81}+\frac{16}{243}+\ldots$, then subtract the second equation from the first to yield $\frac{2}{3}S=\frac{4}{3}+\frac{4}{9}+\frac{4}{27}+\frac{4}{81}+\ldots\Rightarrow\frac{2}{3}S=\frac{4/3}{1-1/3}=\frac{4/3}{2/3}=2\Rightarrow S=3$. |
| 10 | 11 | | Let $S=\sum_{n=1}^{\infty}\frac{F_n}{3^n}=\frac{1}{3}+\frac{1}{9}+\frac{2}{27}+\frac{3}{81}+\frac{5}{243}+\ldots$. Multiply this equation by $\frac{1}{3}$ on both sides to yield $\frac{1}{3}S=\frac{1}{9}+\frac{1}{27}+\frac{2}{81}+\frac{3}{243}+\frac{5}{729}+\ldots$, then subtract the second equation from the first to yield $\frac{2}{3}S=\frac{1}{3}+\frac{1}{27}+\frac{1}{81}+\frac{2}{243}+\frac{3}{729}+\ldots\Rightarrow\frac{2}{3}S=\frac{1}{3}+\frac{1}{9}\left(\frac{1}{3}+\frac{1}{9}+\frac{2}{27}+\frac{3}{81}+\ldots\right)$ $\Rightarrow\frac{2}{3}S=\frac{1}{3}+\frac{1}{9}S\Rightarrow\frac{5}{9}S=\frac{1}{3}\Rightarrow S=\frac{1}{3}\cdot\frac{9}{5}=\frac{3}{5}$. |
| 11 | 12 | 12 | Since $y, x, z$ is arithmetic and $x+y+z=24$, $x=8$. Let $y=8-d$ and $z=8+d$, then $\frac{8-d}{8}=\frac{8+d}{8-d}\Rightarrow 64-16d+d^2=64+8d\Rightarrow 0=d^2-24d=d(d-24)\Rightarrow d=0$ or $d=24$. Since the terms were distinct, $d=24$, and thus $y=8-24=-16$. |
| | | 13 | $\frac{a_{n+1}}{a_n}=\frac{\binom{n+2}{2}}{\binom{n+1}{2}}=\frac{(n+2)!(2)!(n-1)!}{(n+1)!(2)!(n)!}=\frac{n+2}{n}=1+\frac{2}{n}$. Since $\frac{2}{n}$ gets smaller as *n* increases, the greatest value of $\frac{2}{n}$ occurs when *n* is least, namely $n=2$. Therefore, the greatest such value is $1+\frac{2}{2}=1+1=2$. |
| 12 | 13 | | $\sqrt{3}=\sum_{n=0}^{\infty}\left(\cos\theta\cdot\sin^n\theta\right)=\frac{\cos\theta}{1-\sin\theta}\Rightarrow\cos\theta=\sqrt{3}(1-\sin\theta)\Rightarrow\cos^2\theta=3\left(1-2\sin\theta+\sin^2\theta\right)$ $\Rightarrow 1-\sin^2\theta=3-6\sin\theta+3\sin^2\theta\Rightarrow 0=4\sin^2\theta-6\sin\theta+2=2(2\sin\theta-1)(\sin\theta-1)$ $\Rightarrow\sin\theta=1$ or $\sin\theta=\frac{1}{2}$. Since the ratio of the geometric series was $\sin\theta$, we must have $\sin\theta=\frac{1}{2}$, and the smallest positive value of $\theta$ is $\theta=\frac{\pi}{6}$. |
| 13 | 14 | 14 | Since $a_1=2013$, $a_2=2^2+0^2+1^2+3^2=4+0+1+9=14$. Finding the remaining terms of the sequence in a similar manner, the sequence is 2013, 14, 17, 50, 25, 29, 85, 89, 145, 42, 20, 4, 16, 37, 58, 89,..., so the first repeated term is 89. |
| 14 | 15 | 15 | There are a total of $5^5=3125$ distinct sequences, and the number of sequences were there are no repeating terms is $5!=120$. Therefore, the number of sequences with at least one digit repeated is $3125-120=3005$. |

| | | | |
|---|---|---|---|
| 15 | | | $\frac{\pi^2}{6}=\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\ldots$, and let $X=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=\frac{1}{1}-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\ldots$. <br> Subtract the second equation from the first equation to yield $\frac{\pi^2}{6}-X$ <br> $=2\left(\frac{1}{4}+\frac{1}{16}+\frac{1}{36}+\ldots\right)=\frac{2}{4}\left(\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\ldots\right)=\frac{1}{2}\cdot\frac{\pi^2}{6}=\frac{\pi^2}{12}\Rightarrow X=\frac{\pi^2}{6}-\frac{\pi^2}{12}=\frac{\pi^2}{12}$. |