2017 – 2018 Log1 Contest Round 2

Theta Number Theory

Name: _____

Units do not have to be included.

4 points each		
1	What is the greatest common factor, GCD, of 51 and 119?	
2	What is the units digit of 3^{2017}?	
3	If the 6^{th} term of an arithmetic sequence is 7 and the 12^{th} term is 11, what is the 15^{th} term?	
4	How many non-congruent rectangles whose length and width are integers have their area equal to 2 times their perimeter in magnitude?	
5	What is the smallest positive integer that has a remainder of 1 when divided by 11 and a remainder of 2 when divided by 13?	

5 points each		
6	What is the product of the four-smallest positive prime numbers that are congruent to 2 mod 5?	
7	How many positive even factors does the number 1456 have?	
8	How many distinct prime factors does the number 9 factorial have?	
9	How many 2-digit numbers have the digit 3 in them?	
10	The sum of the first n positive perfect cubes is a multiple of 23. What is the smallest value for n?	

6 points each		
11	How many even two-digit numbers have an odd number of positive factors?	
12	Three pairwise relatively prime (no common factors other than 1) positive numbers greater than 1 have a product of 2520. How many distinct ways can these numbers be chosen?	
13	What is the sum of the positive prime numbers that are less than 100 and are 1 greater than a multiple of 3 and 1 less than a multiple of 5?	
14	What is the largest integer that divides $n^4 + 2n^3 - n^2 - 2n$ for every integer n that does not set the expression equal to zero?	
15	Jack bought a gift card for his mother Joan. Being an avid mathematician, he noted that every set of three consecutive digits in the 14-digit card number added to 19. Two of the digits given below are 5 and 9, what is the missing digit 'X'? 	

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1	What is the greatest common factor, GCD, of 51 and 119?	
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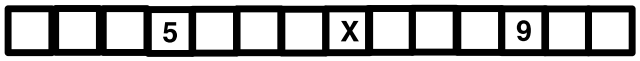
3	If the 6^{th} term of an arithmetic sequence is 7 and the 12^{th} term is 11, what is the 15^{th} term?	
4	How many two-digit positive integers ending in 3 are prime numbers?	
5	What is the smallest positive integer that has a remainder of 1 when divided by 11 and a remainder of 2 when divided by 13?	

5 points each		
6	What is the product of the four-smallest positive prime numbers that are congruent to 2 mod 5?	
7	How many positive even factors does the number 1456 have?	
8	How many distinct prime factors does the number 9 factorial have?	
9	How many positive integers less than 120 are relatively prime to 120?	
10	The sum of the first n positive perfect cubes is a multiple of 23. What is the smallest value for n?	
6 points each		

11	How many even two-digit numbers have an odd number of positive factors?	
12	Three pairwise relatively prime (no common factors other than 1) positive numbers greater than 1 have a product of 2520. How many distinct ways can these numbers be chosen?	
13	What is the sum of the positive prime numbers that are less than 100 and are 1 greater than a multiple of 3 and 1 less than a multiple of 5?	
14	Write the integers from 11 to 30 inclusive back to back, 11121314 … 2930 creating a 40-digit number. What is the remainder when this number is divided by 72?	
15	Jack bought a gift card for his mother Joan. Being an avid mathematician, he noted that every set of three consecutive digits in the 14-digit card number added to 19. Two of the digits given below are 5 and 9, what is the missing digit 'X'? 	

2017 – 2018 Log1 Contest Round 2

Mu Number Theory

Name: _____

Units do not have to be included.

4 points each		
1	What is the greatest common factor, GCD, of 51 and 119?	
2	What is the units digit of 3^{2017}?	
3	If the 6^{th} term of an arithmetic sequence is 7 and the 12^{th} term is 11, what is the 15^{th} term?	
4	How many two-digit positive integers ending in 3 are prime numbers?	
5	Find the sum of the smallest 50 positive integers that are not perfect squares.	

5 points each		
6	What is the product of the four-smallest positive prime numbers that are congruent to 2 mod 5?	
7	How many positive even factors does the number 1456 have?	
8	How many distinct prime factors does the number 9 factorial have?	
9	How many positive integers less than 120 are relatively prime to 120?	
10	How many 3-digit numbers have the digit 3 in them?	

6 points each		
11	How many even two-digit numbers have an odd number of positive factors?	
12	Three pairwise relatively prime (no common factors other than 1) positive numbers greater than 1 have a product of 2520. How many distinct ways can these numbers be chosen?	
13	What is the sum of the positive prime numbers that are less than 100 and are 1 greater than a multiple of 3 and 1 less than a multiple of 5?	
14	Write the integers from 11 to 30 inclusive back to back, 11121314 … 2930 creating a 40-digit number. What is the remainder when this number is divided by 72?	
15	Given three positive integers, a, b, c such that: ab is divisible by 16; bc is divisible by 18 and ac is divisible by 15. What is the largest integer that must divide into abc evenly?	

2017 – 2018 Log1 Contest Round 2

Theta Number Theory – Answer Key

Name: _____

Units do not have to be included.

4 points each		
1	What is the greatest common factor, GCD, of 51 and 119?	17
2	What is the units digit of 3^{2017}?	3
3	If the 6^{th} term of an arithmetic sequence is 7 and the 12^{th} term is 11, what is the 15^{th} term?	13
4	How many non-congruent rectangles whose length and width are integers have their area equal to 2 times their perimeter in magnitude?	3
5	What is the smallest positive integer that has a remainder of 1 when divided by 11 and a remainder of 2 when divided by 13?	67

5 points each		
6	What is the product of the four-smallest positive prime numbers that are congruent to 2 mod 5?	8806
7	How many positive even factors does the number 1456 have?	16
8	How many distinct prime factors does the number 9 factorial have?	4
9	How many 2-digit numbers have the digit 3 in them?	18
10	The sum of the first n positive perfect cubes is a multiple of 23. What is the smallest value for n?	22

6 points each		
11	How many even two-digit numbers have an odd number of positive factors?	3
12	Three pairwise relatively prime (no common factors other than 1) positive numbers greater than 1 have a product of 2520. How many distinct ways can these numbers be chosen?	6
13	What is the sum of the positive prime numbers that are less than 100 and are 1 greater than a multiple of 3 and 1 less than a multiple of 5?	98
14	What is the largest integer that divides $n^4 + 2n^3 - n^2 - 2n$ for every integer n that does not set the expression equal to zero?	24
15	Jack bought a gift card for his mother Joan. Being an avid mathematician, he noted that every set of three consecutive digits in the 14-digit card number added to 19. Two of the digits given below are 5 and 9, what is the missing digit 'X'? 	5

2017 – 2018 Log1 Contest Round 2

Alpha Number Theory – Answer Key

Name: _____

Units do not have to be included.

	4 points each	
1	What is the greatest common factor, GCD, of 51 and 119?	17
2	What is the units digit of 3^{2017}?	3
3	If the 6^{th} term of an arithmetic sequence is 7 and the 12^{th} term is 11, what is the 15^{th} term?	13
4	How many two-digit positive integers ending in 3 are prime numbers?	6
5	What is the smallest positive integer that has a remainder of 1 when divided by 11 and a remainder of 2 when divided by 13?	67

	5 points each	
6	What is the product of the four-smallest positive prime numbers that are congruent to 2 mod 5?	8806
7	How many positive even factors does the number 1456 have?	16
8	How many distinct prime factors does the number 9 factorial have?	4
9	How many positive integers less than 120 are relatively prime to 120?	32
10	The sum of the first n positive perfect cubes is a multiple of 23. What is the smallest value for n?	22
	6 points each	

11	How many even two-digit numbers have an odd number of positive factors?	3
12	Three pairwise relatively prime (no common factors other than 1) positive numbers greater than 1 have a product of 2520. How many distinct ways can these numbers be chosen?	6
13	What is the sum of the positive prime numbers that are less than 100 and are 1 greater than a multiple of 3 and 1 less than a multiple of 5?	98
14	Write the integers from 11 to 30 inclusive back to back, 11121314 … 2930 creating a 40-digit number. What is the remainder when this number is divided by 72?	50
15	Jack bought a gift card for his mother Joan. Being an avid mathematician, he noted that every set of three consecutive digits in the 14-digit card number added to 19. Two of the digits given below are 5 and 9, what is the missing digit 'X'? 	5

2017 – 2018 Log1 Contest Round 2

Mu Number Theory – Answer Key

Name: _____

Units do not have to be included.

4 points each		
1	What is the greatest common factor, GCD, of 51 and 119?	17
2	What is the units digit of 3^{2017}?	3
3	If the 6^{th} term of an arithmetic sequence is 7 and the 12^{th} term is 11, what is the 15^{th} term?	13
4	How many two-digit positive integers ending in 3 are prime numbers?	6
5	Find the sum of the smallest 50 positive integers that are not perfect squares.	1513

5 points each		
6	What is the product of the four-smallest positive prime numbers that are congruent to 2 mod 5?	8806
7	How many positive even factors does the number 1456 have?	16
8	How many distinct prime factors does the number 9 factorial have?	4
9	How many positive integers less than 120 are relatively prime to 120?	32
10	How many 3-digit numbers have the digit 3 in them?	252

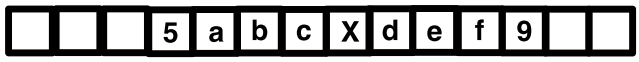
6 points each		
11	How many even two-digit numbers have an odd number of positive factors?	3
12	Three pairwise relatively prime (no common factors other than 1) positive numbers greater than 1 have a product of 2520. How many distinct ways can these numbers be chosen?	6
13	What is the sum of the positive prime numbers that are less than 100 and are 1 greater than a multiple of 3 and 1 less than a multiple of 5?	98
14	Write the integers from 11 to 30 inclusive back to back, 11121314 … 2930 creating a 40-digit number. What is the remainder when this number is divided by 72?	50
15	Given three positive integers, a, b, c such that: ab is divisible by 16; bc is divisible by 18 and ac is divisible by 15. What is the largest integer that must divide into abc evenly?	720

2017 – 2018 Log1 Contest Round 2
Number Theory Solutions

Mu	Al	Th	Solution
1	1	1	Express these numbers in factor form. $51=3(17)$ and $119=7(17)$. It is obvious that 17 is the GCD
2	2	2	List the powers of 3 and determine the power where the units-digit begins to repeat $3^1 = 3, 3^2 = 9, 3^3 = 27, 3^4 = 81,$ and $3^5 = 243$. Every power that is a multiple of 4 ends with a "1". 2017 has a remainder of 1 when divided by 4, so the units-digit is 3.
3	3	3	One could find the common difference, $\frac{12-6}{4}$, but it is easier to find the common difference between every 3 terms $= \frac{4(3)-2(3)}{2(2)} = \frac{3}{2}$ so the 15^{th} term is $11+2=13$
4	4		Consider the numbers that end in 3 for each "decade" of numbers that are 2 digits. 13, 23, **33**, 43, 53, **63**, 73, 83, and **93**. There are 9 such numbers that end in 3 but only 6 of these are prime.
		4	Call the integers m and n. Then $mn = 4(m + n)$. Express this in terms of m. $$mn = 4m + 4n$$ $$m(n-4) = 4n$$ $$m = \frac{4n}{n-4}$$ $$m = \frac{4(n-4)}{n-4} + \frac{16}{n-4}$$ $$m = 4 + \frac{16}{n-4}$$ This means that $n-4$ divides 16. Therefore $n-4 = 1, 2, 4, 8$ or 16. Using the equation above gives the solutions $(5, 20), (6, 12)$ and $(8, 8)$. The other two are equivalent to the first two.
5			There are 7 perfect squares less than 50. These are 1, 4, 9, 16, 25, 36, and 49. The sum of these is 140. Since we must not count these 7 numbers, then it is required to find the sum of the first 57 positive integers and subtract 140 from the answer. $$\text{Sum} = \frac{57(58)}{2} - \frac{7(8)(2(7)+1)}{6}$$ $$\text{Sum} = 29(57) - 7(20)$$ $$\text{Sum} = 1653 - 140 = 1513.$$

	5	5	For integers p and q, $$x = 11p + 1 = 13q + 2.$$ Equate and solve for p. $p = \frac{13q+1}{11} = q + \frac{2q+1}{11}$. By inspection, the smallest value for q that makes p an integer is 5. Therefore, if $q = 5$, then $p = 6$. The number is then $11(6) + 1 = 67$.
6	6	6	Set up the congruence relation. $$x \equiv 2 \bmod 5$$ This is equivalent to all possible numbers that have a remainder of 2 when divided by 5. Mathematically, $$x = 5k + 2; k \in \{non - negative\ integers\}$$ Listing the values of x, $$x = \{2, 7, 12, 17, 22, 27, 32, 37, \ldots.\}$$ The 4-smallest positive primes are 2, 7, 17, and 37. Multiplying gives $2(7)(17)(37) = 8806$
7	7	7	Since $1456 = 2^4 * 7 * 13$, it has $(5)(2)(2) = 20$ positive factors. Of these factors, the only ones that are odd are 1, 7, 13, and 91 (i. e. $7 * 13$). Subtract 4 from 20 to get 16 even positive factors.
8	8	8	In factor form, $9! = 9 * 8 * 7 * 6 * 5 * 4 * 3 * 2$ $$9! = 3^2 * 2^3 * 7 * 2 * 3 * 5 * 2^2 * 3 * 2 = 2^7 * 3^4 * 5 * 7$$ It is obvious that the only prime factors are 2, 3, 5, and 7 so there are 4 of these.
9	9		$120 = 2^3(3)(5)$. One-half of the numbers are not divisible by 2, two-thirds are not divisible by 3 and four-fifths are not divisible by 5 so the total number of relatively prime numbers is: $120\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(\frac{4}{5}\right) = 32$.
		9	**1st method:** Just count them. Answer equals 18. **2nd method:** Count them using combinatorics. There are 90 two-digit numbers. The possible numbers that do not have a 3 in them require that there be 8 numbers to choose for the first digit and 9 to choose for the second digit. Thus there are $8(9) = 72$ digits which do not have a 3. Subtract from 90 to get 18 digits that have a 3.

10			**Method 1:** Just count them. The 100s – 900s, exclusive of the 300s, each have 19 numbers with a 3 in them since we must additionally count, 103, 203, 403, etc. The 300s all have a 3 in them. The total number of 3 digit numbers with a 3 in them is $19(8)+100=252$ **Method 2:** In the same manner of the Theta solution for #9: $900 - 8(9)(9) = 252$
	10	10	The sum of the first n cubes is: $\left[\frac{n(n+1)}{2}\right]^2 = 23q$, where $q \in \{\text{positive integers}\}$ This means $23 \mid \frac{n(n+1)}{2}$. The smallest value for n must be 22.
11	11	11	To have an odd number of factors, it must be a perfect square. The values are 16, 36, and 64.
12	12	12	Express 2520 in factor form. $2520 = 2^3(3^2)(5)(7)$. There are at most 4 pairwise relatively prime numbers that multiply to 2520: 5, 7, 8, and 9. To obtain 3 pairwise relatively prime numbers that multiply to 2520, just multiply any two of the four and let the other two stand-alone. For example, $35(8)(9) = 2520$ or $5(56)(9) = 2520$. With this method, there are 4C2 distinct ways to form 3 number sets. Thus, the answer is 6.
13	13	13	**Method 1: Brute force.** For numbers 1 less than 5k: 4,9,14,19,24,29,34,39,44,49,54,59,64,69,74,79,84,89,94, and 99. Of these numbers, the only ones that are 1 greater than a multiple of 3 are 19 and 79. Thus the answer is $19 + 79 = 98$. **Method 2: Analytically** 4 is the first number that satisfies the remainders. Since 3 and 5 are relatively prime, the next will be $4 + 3(5) = 19$. The others are 34, 49, 64, 79 and 94. Therefore $19 + 79 = 98$.
14	14		The remainder when divided by 8 is the same as remainder of 930 divided by 8 which is 2. The sum of the digits is $\text{Sum} = 9(1) + 10(2) + 1(3) + 2(0 + 1 + 2 + \cdots + 9)$ $\text{Sum} = 32 + 90 = 122$, which has a remainder of 5 when divided by 9. The same will be true of the remainder of the original number. Of the possible remainders, 0…71, the only number that meets that criteria is 50.

		14	$n^4 + 2n^3 - n^2 - 2n = (n-1)(n)(n+1)(n+2)$, which is the product of 4 consecutive integers. There is at least one multiple of 3 and two even numbers, one of which is a multiple of 4 so the product must be divisible by $(3)(2)(4) = 24$. We choose only those values of n that do not set the expression equal to zero. In that scenario, every number is a divisor and those possibilities are non-sensical.
15			16 | ab so 16 | abc. Also, $18 = 2(9)$| bc so 9 | abc 9 and 16 are relatively prime. $15 = 3(5)$ | ac so 5 | abc. Since 16, 9 and 5 are relatively prime and each divides abc so must (16)(9)(5)= 720
	15	15	 Since every three consecutive numbers add to 19, $5 + a + b = 19$ and $a + b + c = 19$ so $c = 5$. Also $e + f + 9 = 19$ and $d + e + f = 19$ so $d = 9$. Finally $c + X + d = 19$ and $X = 5$.