**Question #1**

\[9x^2 + 16y^2 + 18x = 64y + 71 \rightarrow 9x^2 + 18x + 16y^2 - 64y = 71 \rightarrow\]

\[9(x^2 + 2x + 1) + 16(y^2 - 4y + 4) = 71 + 9 + 64 = 144 \rightarrow \frac{(x + 1)^2}{16} + \frac{(y - 2)^2}{9} = 1\]

\[a = \sqrt{16} = 4, \ b = \sqrt{9} = 3\]

\[A = \text{area} = ab\pi = 12\pi\]

\[c^2 = a^2 - b^2 \rightarrow c = \pm\sqrt{7}\]

Foci: \((-1\pm\sqrt{7}, 2)\)

\[B = \text{sum of abscissas} = (-1 + \sqrt{7}) + (-1 - \sqrt{7}) = -2\]

Center = \((-1, 2)\)

\[C = \text{product of abscissa and ordinate:} \ -1 \cdot 2 = -2\]

\[D = \text{eccentricity} = \frac{c}{a} = \frac{\sqrt{7}}{4}\]

\[A \cdot B \cdot C \cdot D = 12\pi \cdot -2 \cdot -2 \cdot \frac{\sqrt{7}}{4} = 12\pi\sqrt{7}\]

**Question #2**

\[R = 10(20 + 0) - 3(-5 - 0) + 2(1 - 28) = 161\]

If only one column is multiplied by 2, then the original determinant is doubled: \(S = 2\)

\[8^n = 2^{3n}\] will divide \(44^{44} = 2^{88} \cdot 11^{44}\) only when \(3n \leq 88\). The largest \(n\) is therefore 29, \(T\).

Congruent triangles are formed by SSS.

\[x^2 = 21^2 + 28^2\]
\[= 441 + 784\]
\[= \sqrt{1225}\]
\[x = 35\]

\[P = 35 + 35 + 56\]
\[= 126\]

\[R + S + T + U = 161 + 2 + 29 + 126 = 318\]
Question #3

\[ \frac{A+3}{4} + \frac{B-1}{3} = 1 \quad \rightarrow \quad \frac{3A + 9 + 4B - 4}{2A - B} = 12 \quad \rightarrow \quad \frac{3A + 4B}{2A - B} = 12 \]

\[ \begin{align*}
(A, B) &= (5, -2) \\
\frac{12}{C} - \frac{12}{D} &= 7 \\
\frac{3}{C} + \frac{4}{D} &= 0
\end{align*} \]

Let \( x = \frac{1}{C} \), \( y = \frac{1}{D} \)

\[ \begin{align*}
12x - 12y &= 7 \\
3x + 4y &= 0
\end{align*} \]

\[ \begin{align*}
(C, D) &= (3, -4)
\end{align*} \]

\[ \sin x \cos E + \cos x \sin E = \sin(x + E) \rightarrow \sin(22 + E) = \frac{\sqrt{3}}{2} \]

The angle with the required sine is \( 60^\circ \), so \( E = 38^\circ \).

\( ABCDE = 5 \cdot -2 \cdot 3 \cdot -4 \cdot 38 = 4560 \)

Question #4

\[ \left( x^3 + 1 \right)^4 \left( x^4 + 1 \right)^5 = \left( x^{12} + \ldots \right) \left( x^{20} + \ldots \right) = x^{32} + \ldots \]

The degree is 32, \( A \).

\[ 7 \big| 298 \]

\[ \begin{array}{c|cc}
42 & R4 \\
6 & R0
\end{array} \]

\[ 298_{10} = 604_{7} \quad B = 604. \]

\[ 320 = 20 + (5 - 1)d \rightarrow 300 = 4d \rightarrow d = 75 \rightarrow a_{12} = 320 + 2(75) = 470, \]

\[ \gamma P3 = 7 \cdot 6 \cdot 5 = 210 \quad \frac{1}{6!} = \frac{1}{720} \quad 4C2 = \frac{4!}{2!2!} = 6 \quad 4P2 = 4 \cdot 3 = 12 \]

Diagonal matrix, so determinant = product of diagonal entries: \( 210 \cdot \frac{1}{720} \cdot 6 \cdot 12 = 21 = D \)

\[ B - A - C - D = 604 - 32 - 470 - 21 = 81. \]
Question #5

\[ \begin{align*} 
(2 + m)(2 + n) - 5(2) + n &= 0 \\
(1 + m)(1 + n) - 5(-1) + n &= 0 
\end{align*} \]

\[ \begin{align*} 
24 + 4m - 10 + n &= 0 \\
3 + m + 5 + n &= 0 \\
4m + n &= -14 \\
m + n &= -2 
\end{align*} \]

\[ A = mn = -8 \]

\[ B = (-1 + i)^4 = \left[ \sqrt{2} \left( \cos 135^\circ + i \sin 135^\circ \right) \right]^4 = \frac{1}{\left( \sqrt{2} \right)^4} \left[ \cos(-540^\circ) + i \sin(-540^\circ) \right] \]

\[ - \frac{1}{4} (-1 + 0i) = -\frac{1}{4} \]

\[ ME = \frac{1}{2} CD \] because it is a median of triangle \( ADC \).

\[ NF = \frac{1}{2} CD \] because it is a median of triangle \( BCD \).

\[ MN = \frac{1}{2} (CD + AB) \] because it is a median of the trapezoid.

\[ MN = \frac{1}{2} (8 + 20) = 14; \quad ME = \frac{1}{2} (8) = 4 = NF; \quad 14 - 4 - 4 = 6 = C \]

If the vertical asymptote is at 4, then \( c = 4 \). If horizontal asymptote is 2, then \( a = 2 \) because leading coefficient on bottom is 1. If \( y \)-intercept is 1, then \( 1 = \frac{2(0) + b}{(0) - 4} \rightarrow -4 = b \). \( D = \) sum of 2, -4, 4 = 2.

\[ A \bullet B \bullet C \bullet D = -8 - \frac{1}{4} \bullet 6 \bullet 2 = 24 \]

Question #6

\[ \begin{align*} 
\log_3 4 + \log_3 8 + \log_3 16 + \log_3 32 &= \frac{\log_3 4}{\log_3 2} + \frac{\log_3 8}{\log_3 2} + \frac{\log_3 16}{\log_3 2} + \frac{\log_3 32}{\log_3 2} \\
&= \log_2 4 + \log_2 8 + \log_2 16 + \log_2 32 = 2 + 3 + 4 + 5 = 14 = A. 
\end{align*} \]

Semiperimeter of quadrilateral = \[ \frac{1 + 5 + 9 + 11}{2} = 13 \]

By Brahmagupta’s formula, area = \( \sqrt{(13 - 1)(13 - 5)(13 - 9)(13 - 11)} = \sqrt{768} = B. \quad B^2 = 768. \)

\[ \frac{\sqrt{0.005}}{\sqrt{2}} = \sqrt{\frac{5}{1000}} = \sqrt{\frac{5}{2000}} = \sqrt{\frac{1}{400}} = \frac{1}{20} = 5\%. \quad C = 5. \]

After obtaining the common denominator and simplifying, \[ \frac{1}{3} \leq \frac{x}{2007} \leq \frac{4}{9} \quad \rightarrow \quad 669 \leq D \leq 892. \]

The number of integers is \( 892 - 669 + 1 = 224, D. \)

The largest digit from \( A, B, C, \) and \( D \) is 8.
Question #7

\[ x^7 + 125x^4 - x^3 - 125 \rightarrow x^4(x^3 + 125) - 1(x^3 + 125) \rightarrow (x^4 - 1)(x^3 + 125) \]
\[ \rightarrow (x^2 + 1)(x^2 - 1)(x + 5)(x^2 - 5x + 25) \]
\[ \rightarrow (x^2 + 1)(x^2 - 5x + 25)(x - 1)(x + 1)(x + 5) \]

There are three linear factors in the factorization, so \( A = 3 \).

\[ 4^B - 4^{B-1} = 24 \rightarrow 4^B(1 - 4^{-1}) = 24 \rightarrow 4^B\left(\frac{3}{4}\right) = 24 \rightarrow 4^B = 32 \rightarrow 2^{2B} = 2^5 \]
\[ B = \frac{5}{2} \]

\[ |3x + 7| ≤ 1(=5^0) \rightarrow 3x + 7 \leq 1 \text{ and } 3x + 7 \geq -1 \left[ \frac{-8}{3}, -2 \right]. \]

-2 is the smallest integer in the solution set, \( C \).

\[ A \cdot B \cdot C = 3 \cdot \frac{5}{2} \cdot -2 = -15. \]

Question #8

The prime factorization of 8400 = 2^4 \cdot 3 \cdot 5^2 \cdot 7. If a perfect square divides this product, it can have only 2 and 5 among its factors. It can only have zero, two, or four 2s and only zero or two 5s. Since the number is square it cannot have an odd number of any factor. Therefore, the number of ways is (3)(2) = 6.

\[ \frac{x + 7}{x^2 - x - 6} = \frac{B}{x - 3} + \frac{C}{x + 2} = \frac{B(x + 2)}{(x - 3)(x + 2)} + \frac{C(x - 3)}{(x - 3)(x + 2)} \]
\[ \rightarrow x + 7 = B(x + 2) + C(x - 3) \]
If \( x = 3 \): \( 3 + 7 = B(3 + 2) + C(0) \rightarrow 10 = 5B \rightarrow B = 2 \)
If \( x = -2 \), \( -2 + 7 = B(0) + C(-2 - 3) \rightarrow 5 = -5C \rightarrow C = -1. \)

\( A = 6, B = 2, C = -1 \rightarrow 6 \div 2 \div -1 = -3. \)

Question #9

\[ \begin{cases} y = 3x^2 - 2x + 5 \\ y = 4x + 2 \end{cases} \rightarrow 3x^2 - 2x + 5 = 4x + 2 \]
\[ 3x^2 - 6x + 3 = 0 \rightarrow x^2 - 2x + 1 = 0 \rightarrow (x - 1)^2 = 0 \]
\( x = 1 \)
\( y = 4(1) + 2 = 6 \)

Intersection: (1, 6) Distance = \( \sqrt{(3 - 1)^2 + (17 - 6)^2} = \sqrt{4 + 121} = \sqrt{125} = A \). \( A^2 = 125. \)

181 is the sum of two squares. 100 and 81 are fairly obvious. Checking with 10 and 9, we find that this pair works so \( ab = 90. \) Also, \( (a + b)^2 = a^2 + 2ab + b^2 = 19^2 = 361. \) \( 2ab = 361 - 181 = 180, \) so \( ab = 90, B. \)

\( 64 = 2^6 \) so it is a sixth, a cube, a square, and a first power.
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\[
\begin{align*}
&z = 6 : (x, y) = (2, 1) \\
&z = 3 : (x, y) = (4, 1), (2, 2) \\
&z = 2 : (x, y) = (8, 1), (2, 3) \\
&z = 1 : (x, y) = (64, 1), (8, 2), (4, 3), (2, 6)
\end{align*}
\]

There are \( C = 9 \) possible triples.

The units digits to the above answers are 5, 0, and 9. Their sum is 14.

**Question #10**

When evaluated at \( x = 1 \),

\[
\lim_{x \to 1} \frac{\sqrt[3]{x} - 1}{x - 1} = \lim_{x \to 1} \frac{\sqrt[3]{x} - 1}{\sqrt[3]{x^2} + \sqrt[3]{x} + 1}.
\]

Therefore, \( \lim_{x \to 1} \frac{\sqrt[3]{x} - 1}{x - 1} = \lim_{x \to 1} \frac{1}{\sqrt[3]{x^2} + \sqrt[3]{x} + 1} \). Evaluating again at \( x = 1 \), \( \frac{1}{3} = \frac{A}{B} \).

\[
\sqrt{PE \cdot ET} = 16 \to (PE)(ET) = 256.
\]

**Area** = \( 84 = \frac{1}{2} (PE)(ET) \sin P \)

\[
\sin P = \frac{21}{32} = C
\]

1 + 3 + 2 + 1 + 3 + 2 = 12

**Question #11**

\[
\log_{\cos x} \sin x = \frac{1}{2} \to (\cos x)^{\frac{1}{2}} = \sin x
\]

\[
\cos x = \sin^2 x \to \cos x = 1 - \cos^2 x
\]

\[
\cos^2 x + \cos x - 1 = 0
\]

\[
[\cos x] = \left[ \frac{-1 \pm \sqrt{1 - 4(1)(-1)}}{2(1)} \right] = \left[ \frac{-1 + \sqrt{5}}{2} \right] = A \approx \left[ \frac{-1 + 2.2}{2} \right] = 0
\]

The other value of \( \cos x \) is extraneous since \( x \) must be in Quadrant I.

Using sum and product of roots, product \( = \sin u \cos u = \frac{k}{9} \) and sum \( = \sin u + \cos u = \frac{2}{9} \).

\[
\sin u + \cos u = \frac{2}{9} \to (\sin u + \cos u)^2 = \sin^2 u + \cos^2 u + 2 \sin u \cos u = \frac{4}{81}
\]

\[
= 1 + 2 \left( \frac{k}{9} \right) = \frac{4}{81}; k = -\frac{77}{18}
\]

\[
A + B = 0 - \frac{77}{18} = -\frac{77}{18}
\]
Question #12

\( e \approx 2.72, \pi \approx 3.14 \quad \Rightarrow \quad e + \pi \approx 5.86 \Rightarrow 6. \)

There are fewer 5’s in 40! than 2’s, so divide:

\[
\begin{array}{c}
5 \div 40 \\
5 \div 8 \\
8+1 = 9 = B
\end{array}
\]

\[
x^3 + x - 8 = \frac{8}{x^2} \quad \Rightarrow \quad x^5 + x^3 - 8x^2 - 8 = 0 \quad \Rightarrow \quad x^3(x^2 + 1) - 8(x^2 + 1) = 0
\]

\[
\Rightarrow (x^2 + 1)(x^3 - 8) = 0 \quad \Rightarrow \quad (x^2 + 1)(x - 2)(x^2 + 2x + 4) = 0
\]

\( x^2 + 1 \) and \( x^2 + 2x + 4 \) only yield only imaginary solutions. There is only one real solution, so \( C = 2. \)

\[
\begin{align*}
x^2 + e^2 &= 10^2 \\
y^2 + e^2 &= 8^2 \ \Rightarrow \quad 100 - x^2 = 64 - y^2
\end{align*}
\]

\[
\begin{align*}
x^2 + f^2 &= 11^2 \\
y^2 + f^2 &= z^2 \ \Rightarrow \quad 121 - x^2 = z^2 - y^2
\end{align*}
\]

\[
\Rightarrow 21 = z^2 - 64 \Rightarrow z = \sqrt{85}. \ D = 9
\]

The answer choices are 6, 9, 2, and 9. The mode is 9.

Question #13

The common ratio for each move is \(-\frac{1}{9}\), as the first south move is two moves after the first north move (likewise for east-west) and each move is in the opposite direction as the move two steps before. The first east-west move is 60 m. Therefore, the total distance traveled each direction is an infinite geometric series.

\[
\begin{align*}
\text{North-South:} & \quad \frac{180}{1 - \left(-\frac{1}{9}\right)} = 162 \\
\text{East-West:} & \quad \frac{60}{1 - \left(-\frac{1}{9}\right)} = 54
\end{align*}
\]

The final stop for the armadillo is the point \((54, 162) = (A, B)\).

\( x^{\log x} = 100x \quad \Rightarrow \quad \log x^{\log x} = \log 100x \)

\[
\Rightarrow (\log x)^2 = \log 100 + \log x \quad \Rightarrow \quad (\log x)^2 - \log x - 2 = 0
\]

\[
\Rightarrow (\log x - 2)(\log x + 1) = 0 \quad \Rightarrow \quad \log x = 2, \log x = -1
\]

\( x = 100, \ x = \frac{1}{10} \quad \Rightarrow \quad C = 100 \)

\[
\frac{B - A}{C} = \frac{162 - 54}{100} = \frac{108}{100} \Rightarrow 108\%
\]
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**Question #14**

\[(a - b)^2 = a^2 - 2ab + b^2\]

\[= \left(\sqrt{7 + 2\sqrt{6}}\right)^2 - 2\left(\sqrt{7 + 2\sqrt{6}}\right)\left(\sqrt{7 - 2\sqrt{6}}\right) + \left(\sqrt{7 - 2\sqrt{6}}\right)^2\]

\[x = \sqrt{7 + 2\sqrt{6}} - \sqrt{7 - 2\sqrt{6}} = a - b.\]

\[= 7 + 2\sqrt{6} - 2\sqrt{49 - 24} + 7 - 2\sqrt{6}\]

\[= 14 - 2\sqrt{25}\]

\[= 14 - 10\]

\[= 4 = A\]

Since 990 = 2 \cdot 3^2 \cdot 5 \cdot 11, B! must contain a factor of 11. Since 11 is prime, 11! is the smallest factorial that contains a factor of 11, so \(B = 11\).

\[A + B = 15.\]

**Question #15**

\[1,000,000,000 = 10^9 = 2^9 \cdot 5^9 = 1,953,125 \cdot 512 \quad \rightarrow \quad A = 512\]

Let \(B = x^2, n = 43\).

\[x^2 + n = y^2 \quad \rightarrow \quad n = y^2 - x^2 = (y + x)(y - x)\]

\[\rightarrow \quad 43 = (y + x)(y - x)\]

This occurs when \(y - x = 1, y + x = 43 \quad \rightarrow \quad x = 21, y = 22\).

\[21^2 + 43 = 22^2 \quad \text{so the desired square number } B \text{ is } B = 21^2 = 441.\]

TAMPA has 5 letters but 2 A’s, so the desired set-up is \(\frac{5!}{2!} = 5 \cdot 4 \cdot 3 = 60\).

\[A + B + C - 1 = 512 + 441 + 60 - 1 = 1012.\]