2007 MAΘ Nationals

Mu Gemini Solutions

1. We reflect $(10,2,6)$ across $y=-3$ to get $(10,-8,6)$

Now we use Pythagorean: $\sqrt{10^2+8^2+6^2} = \sqrt{200} = \boxed{10\sqrt{2}}$

$\boxed{B}$

2. 4-digit: first digit 1–9, 2nd digit 0–9, other two fixed $\to 9 \cdot 10 = 90$

3-digit: first digit 1–9, 2nd digit 0–9 $\to 90$

2-digit: 9

1-digit 9

total: $\boxed{198}$

$\boxed{C}$

3. $$\int_0^{\pi/3} \tan^3 x\, dx = \int_0^{\pi/3} (\sec^2 x - 1)\tan x = \int_0^{\pi/3} \underset{\uparrow\, u}{\tan x}\, \underset{\uparrow\, du}{\underline{\sec^2 x\, dx}} - \int_0^{\pi/3} \tan x\, dx$$

$$= \frac{(\tan x)^2}{2}\Big|_0^{\pi/3} + \ln(\cos x)\Big|_0^{\pi/3} = \frac{(\sqrt{3})^2}{2} - 0 + \ln\left(\frac{1}{2}\right) = \boxed{\frac{3}{2} - \ln 2}$$

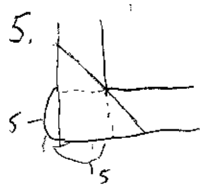
$\boxed{A}$

4. Let's call these a and b

For a given a, we want b to be from 5 to $\frac{50}{a}$

$$\frac{\text{desired} \to \int_5^{10} \left(\frac{50}{a} - 5\right) da}{\text{total} \to \int_5^{10} (10-5)\, da} = \frac{50 \ln a \Big|_5^{10} - 1}{\int_5^{10} 5\, da} = 2\ln 2 - 1 = \boxed{\ln 4 - 1}$$

$\boxed{B}$

5. 

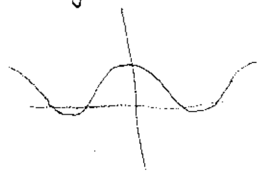
$\sqrt{10^2+10^2+10^2} = \boxed{10\sqrt{3}}$ $\boxed{A}$

6. $V = \pi r^2 h$

$$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt} = t$$

$$\frac{dh}{dt} = \frac{t}{\pi r^2} = \frac{5}{\pi(5^2)} = \boxed{\frac{1}{5\pi}} \quad \boxed{D}$$

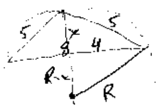
7. $x=0$ is clearly a solution.
Now note that the integrand is negative only very slightly (> -0.1) and for a very short duration.



So the overall area starting from 0 will never again be 0 → no more solutions.

$\boxed{B}$

8.



$x^2+4^2=5^2 \longrightarrow x=3$

$(R-x)^2+4^2=R^2$

$R^2-2Rx+x^2+4^2=R^2$

$2Rx = x^2+4^2$

$R = \boxed{\frac{25}{6}} \quad \boxed{A}$

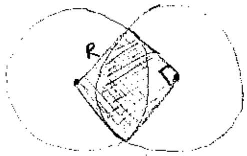
9. Let's call the distance d, escalator speed e, my speed m

$$\frac{d}{e+m} = \frac{1}{4}\left(\frac{d}{m-e}\right)$$

$e+m = 4(m-e)$

$5e = 3m \rightarrow m:e = \boxed{5:3} \quad \boxed{E}$

10.



The overlapping region is (2 areas of circle quarter) − (area of square) $= \frac{\pi R^2}{2} - R^2 = \left(\frac{\pi}{2}-1\right)R^2$ $\boxed{B}$

11. When I have unrolled half of the length of it, $\pi (R_{half})^2 = \frac{1}{2}\pi R^2 \rightarrow R_{half} = \frac{R}{\sqrt{2}}$

1 Revolution $= 2\pi r = (\pi\sqrt{2})(R) = 3\pi\sqrt{2}$ inches

$$\left(\frac{10 \text{ inches}}{\text{second}}\right)\left(\frac{\text{revolution}}{3\pi\sqrt{2} \text{ inches}}\right) = \boxed{\frac{5\sqrt{2}}{3\pi}} \quad \boxed{C}$$

12. $y = (x^2)^{x^2}$

$\ln y = x^2 \ln(x^2) = 2x^2 \ln x$

$\frac{dy}{y} = (4x\ln x + 2x)dx = 2x(2\ln x + 1)\,dx$

$\frac{dy}{dx} = 2x(2\ln x + 1)(x^2)^{x^2}$ $\boxed{A}$

13. $\frac{dh}{dt} = (3 - h)$

grows until $h = 3$, then $\frac{dh}{dt} = 0 \rightarrow$ stops growing $\boxed{C}$

14. $f(0) = 0$

$f(x) = \frac{x^2}{2}$ on $0 \le x \le 1$ so that $f''(x) = 1$

$f(x) = \frac{1}{2} + x$ on $1 \le x \le 5$ so that $f'(x) = 1$

$f(5) = 5.5$ $\boxed{C}$

15. $\log_{16} x + \log_4 x = 6$

$3\log_{16} x = \log_{16}(16)^6$

$x = 256$

$2+5+6 = 13$ $\boxed{B}$

$\log_{16} x = y$

$16^y = x$

$4^{2y} = x$

$2y = \log_4 x = 2\log_{16} x$

16. $\frac{1}{1+\sqrt{1+\frac{1}{1+\sqrt{1+\dots}}}} = x$

also $x = \frac{1}{1+\sqrt{1+x}}$

$x + x\sqrt{1+x} = 1$

$(1-x)^2 = x^2(1+x)$

$x^3 + 2x - 1 = 0$ $\boxed{D}$

17. Start : (1 L milk), (1 L water)

Then: (1 L milk, 0.2 L water), (0.8 L water)

Then: 0.8(1 L milk, 0.2 L water), (0.84 L water, 0.2 L milk)

$\boxed{C}$

18. Note that $\sin((x+y)+y) = \sin(x+y)\cos y + \sin(y)\cos(x+y)$

$\sin(x+y)\cos y = \sin(x+y+y) + \sin(x+y-y)$

So the maximum is at same spot as that for $\sin(x+2y)$

$x+2y = \pi/2$

$\boxed{y = \frac{\pi}{4} - \frac{x}{2}}$ $\boxed{A}$

19. . . . . .

We need to split the five dots into three groups (since each of a, b, c must be at least 1)

|. .|. . .   one possible split (1, 3, 4)

.|. . .|.   another possible split (2, 4, 2)

6 places to put lines, 2 distinct lines $\rightarrow 6+5+4+3+2+1 = \boxed{21 \text{ solutions}}$ $\boxed{B}$

20.

Score = $4c - w$;   c = correct $c+b \leq 30$

$c+b+w = 30 \rightarrow w = 30-(c+b)$ b = blank

score = $4c - (30-(c+b))$ w = wrong

$= 5c + b - 30$

Suppose $s_1 = 5c + b$ for exactly 4 sets of (c, b) : $(c_1, b_1), \dots (c_4, b_4)$, $c_1 < c_2 < c_3 < c_4$

$5c_1 + b_1 = s_1$

$5(c_1 - 1) + (b_1 + 5) = s_1$ so $b_1 + 5 > 30 \rightarrow b_1 > 25$

$5(c_4 + 1) + (b_4 - 5) = s_1 \rightarrow b_4 - 5 < 0 \rightarrow b_4 < 5$

if c_1, c_2 work, then so does anything in between $\rightarrow c_2 = c_1 + 1$

$b_4 = b_1 + 15$ so no solutions $\boxed{E}$

21. 

Area of triangle: $\left(\frac{1}{2}\right)(2)(\sqrt{3})$

Area of circle segments: $3\cdot\left(\frac{1}{6}\pi(1)^2\right)$

$\boxed{\sqrt{3}-\frac{\pi}{2}}$ $\boxed{B}$

22. 

$A=\frac{1}{2}s\left(s\frac{\sqrt{3}}{2}\right)$

$\frac{dA}{dt}=\frac{s}{2}\sqrt{3}\,\frac{ds}{dt}$ $\qquad \frac{10\sqrt{3}}{2}=\boxed{5\sqrt{3}}$ $\boxed{D}$

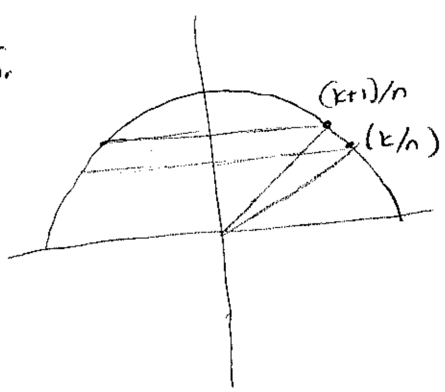
23. $\frac{d}{dx}f^{-1}(x)=\dfrac{1}{\frac{d}{dx}f(f^{-1}(x))}$

$f^{-1}(0)$: $\quad 0=\dfrac{e^{f^{-1}(0)}-e^{-f^{-1}(0)}}{2} \rightarrow f^{-1}(0)=0$

$\frac{df}{dx}=\frac{e^{x}+e^{-x}}{2} \rightarrow \frac{d}{dx}f^{-1}(0)=\dfrac{1}{\left(\frac{e^{0}+e^{-0}}{2}\right)}=1$ $\boxed{C}$

24. $\int\frac{dx}{1-\sin x}=\int\frac{1+\sin x}{(1+\sin x)(1-\sin x)}\,dx=\int\frac{1}{\cos^2 x}+\frac{\sin x}{\cos^2 x}\,dx=\tan x+\sec x\Big|_{\pi/6}^{\pi/3}$

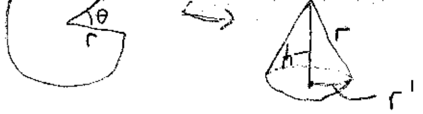
$=\left(\sqrt{3}+2\right)-\left(\frac{\sqrt{3}}{3}+\frac{2\sqrt{3}}{3}\right)=2$ $\boxed{D}$

25. 

$\theta=\arccos\left(\frac{k+1}{n}\right)-\arccos\left(\frac{k}{n}\right)$

arclength $=\theta r=\theta$

There are two of these segments, so $\boxed{C}$



$$2\pi r' = r\left(1-\frac{\theta}{360}\right)$$
$$h = \sqrt{r^2 - r'^2}$$
$$V = \frac{1}{3}\pi r'^2 h = \left(\frac{r\left(1-\frac{\theta}{360}\right)}{2\pi}\right)^2\left(\frac{1}{3}\pi\right) r\sqrt{1-\left(\frac{\left(1-\frac{\theta}{360}\right)}{2\pi}\right)^2}$$

So the θ dependence is $\left(1-\frac{\theta}{360}\right)^2\sqrt{1-\frac{\left(1-\frac{\theta}{360}\right)^2}{2\pi}}$ $\quad u = 1-\frac{\theta}{360}$

$$f(u) = u^2\sqrt{1-\frac{u^2}{2\pi}}$$

$f(u)$ is maximized: $f'(u) = 2u\sqrt{1-\frac{u^2}{2\pi}} - \frac{u^3/2\pi}{\sqrt{1-\frac{u^2}{2\pi}}} = 0$

$$2\left(1-\frac{u^2}{2\pi}\right) = \frac{u^2}{2\pi} \rightarrow u^2 = \frac{2}{\frac{1}{2\pi}+\frac{1}{\pi}} \rightarrow \boxed{\theta = \left(1-\sqrt{\frac{4}{3\pi}}\right)\cdot 360} \quad \boxed{D}$$

27. $I = \int_0^{\pi/6} \sec x \tan^2 x\, dx = \sec x \tan x \Big|_0^{\pi/6} - \int_0^{\pi/6} \sec^3 x\, dx = \sec x \tan x \Big|_0^{\pi/6} - \int_0^{\pi/6} \sec x(1+\tan^2 x)\, dx$

$u = \tan x \quad dv = \sec x \tan x\, dx$

$du = \sec^2 x \quad v = \sec x$

$$2I = \sec x \tan x \Big|_0^{\pi/6} - \int_0^{\pi/6} \sec x\left(\frac{\sec x + \tan x}{\sec x + \tan x}\right) dx = \sec x \tan x \Big|_0^{\pi/6} - \ln(\sec x + \tan x)\Big|_0^{\pi/6}$$

$$I = \frac{1}{2}\left(\frac{2}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}} - 0 - \ln\left(\frac{2}{\sqrt{3}}+\frac{1}{\sqrt{3}}\right) + \ln(1+0)\right) = \boxed{\frac{1}{3} - \frac{\ln 3}{4}} \quad \boxed{A}$$

Probability that red plate #1 is in right place: $\frac{5}{10}$

#2: $\frac{4}{9}$ etc.

$$P = \left(\frac{5}{10}\right)\left(\frac{4}{9}\right)\left(\frac{3}{8}\right)\left(\frac{2}{7}\right)\left(\frac{1}{6}\right) = \boxed{\frac{(5!)^2}{10!}} \quad \boxed{D}$$

$$I = \int_{\pi/3}^{\pi/2} \frac{1}{\sin x + \sqrt{3}\cos x}\, dx$$

$$\sin x + \sqrt{3}\cos x = 2\left(\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x\right) = 2\left(\sin\frac{\pi}{6}\sin x + \cos\frac{\pi}{6}\cos x\right) = \cos\left(x - \frac{\pi}{6}\right)$$

$$I = \frac{1}{2}\int_{\pi/6}^{\pi/3} \frac{1}{\cos x}\, dx = \frac{1}{2}\left(\ln(\sec x + \tan x)\right)\Big|_{\pi/6}^{\pi/3} = \frac{1}{2}\ln\left(\frac{2+\sqrt{3}}{\frac{2}{\sqrt{3}}+\frac{1}{\sqrt{3}}}\right) = \boxed{\frac{1}{2}\ln\left(\frac{2\sqrt{3}+3}{3}\right)} \quad \boxed{A}$$

$$4x + 5y^2 + 6xy = 5$$
$$4dx + 10y\,dy + 6x\,dy + 6y\,dx = 0$$
$$\frac{dy}{dx} = \frac{4+6y}{-(10y+6x)} = \frac{4+6}{10} = \boxed{-1} \quad \boxed{A}$$