1. \( u \cdot v = 2 - 3 + 0 = -1 \). \textbf{A}

2. \( u \times v = \begin{vmatrix} i & j & k \\ 2 & 3 & -5 \\ 1 & -1 & 0 \end{vmatrix} = -5i - 5j - 5k \). \textbf{E}

3. \( u \cdot v = 3 - \sqrt{3} + \sqrt{3} + 1 = 4 \). \( ||u|| = \sqrt{3 + 1} = 2 \). \( ||v|| = \sqrt{3 - 2\sqrt{3} + 1 + 3 + 2\sqrt{3} + 1} = 2\sqrt{2} \).

\[
\cos(\theta) = \frac{4}{2 \cdot 2\sqrt{2}} = \frac{\sqrt{2}}{2}. \quad \theta = 45^\circ. \textbf{C}
\]

4. \( 2A - 3B = \begin{bmatrix} 3 & -10 \\ -11 & 8 \end{bmatrix} \). \( \det(2A - 3B) = 24 - 110 = -86 \). \textbf{E}

5. \[
\cos(\theta) = \frac{x \cdot y}{\|x\| \|y\|} = \frac{\sin(t) + \cos(t)}{\sqrt{2}}. \quad -\sin(\theta)dt = \frac{\cos(t) - \sin(t)}{\sqrt{2}} dt. \quad \frac{d\theta}{dt} = \frac{\sin(t) - \cos(t)}{\sin(\theta) \cdot \sqrt{2}}.
\]

\[
\sin(\theta) = \frac{\|x \times y\|}{\|x\| \|y\|}. \quad x \times y = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 0 & \sin(t) & \cos(t) \end{vmatrix} = (\cos(t) - \sin(t))k. \quad \sin(\theta) = \frac{\|x \times y\|}{\|x\| \|y\|}.
\]

\[
\frac{d\theta}{dt} = \frac{\sin(t) - \cos(t)}{\cos(t) - \sin(t)} = 1 \text{ when } t = \frac{\pi}{3}. \textbf{A}
\]

6. I is true by definition and II follows from I. Since the eigenvalues are the roots of the characteristic polynomial, and since the characteristic polynomial of an \( n \times n \) matrix is an \( n \)th order polynomial, there are \( n \) roots by the Fundamental Theorem of Algebra. Since \( \lambda \) is an eigenvalue of \( A \), there is a vector \( x \) such that \( Ax = \lambda x \). Then \( A^2 x = \lambda A x = \lambda^2 x \) and hence \( \lambda^2 \) is an eigenvalue of \( A^2 \), so IV is true. \textbf{D}

7. A matrix is singular when its determinant is equal to 0. \[
\begin{vmatrix} x & 2 & 0 \\ -1 & 1 & 1 \\ x & 0 & 2 \end{vmatrix} = 4x + 4 = 0. \quad x = -1. \textbf{A}
\]

8. \( \{t, \cos(kt)\} = \int_{-\pi}^{\pi} t \cos(kt) dt \). Notice that \( t \cos(kt) \) is an odd function, so the integral is 0 and hence \( a_k = 0 \). \textbf{B}

9. \( \sin(\theta) = \frac{||u \times v||}{||u|| ||v||} \) and \( \cos(\theta) = \frac{u \cdot v}{||u|| ||v||} \), so \( \frac{||u \times v||^2}{||u||^2 ||v||^2} + \frac{(u \cdot v)^2}{||u||^2 ||v||^2} = 1 \). Therefore \( (u \cdot v)^2 + ||u \times v||^2 = ||u||^2 ||v||^2 \). \textbf{C}

10. Adding a scalar multiple of one row to another row does not alter a matrix’s determinant. Multiplying a row by a scalar multiple increases a matrix’s determinant by a factor of the scalar. The determinant of the new matrix is therefore 12 \( \cdot 5 = 60 \). \textbf{C}
11. \( \text{det}(A) = \begin{vmatrix} 4x+1 & 5-2x \\ 2-2x & x-2 \end{vmatrix} = (4x+1)(x-2) - (5-2x)(2-2x) = 7x - 12. \)

\( A^{-1} = \begin{bmatrix} x-2 & 2-5x \\ 7x-12 & 7x-12 \\ 2x-2 & 4x+1 \\ 7x-12 & 7x-12 \end{bmatrix}. \lim_{x \to \infty} A^{-1} = \begin{bmatrix} 1 \\ 7 \\ 2 \\ 7 \end{bmatrix}. \) The largest entry is \( \frac{4}{7}. \) C

12. Notice that after \( t \) seconds the angle between \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) is \( \frac{\pi}{4} + \frac{\pi}{32} t. \) Geometrically, \( \|\mathbf{v}_1 - \mathbf{v}_2\| \) is the distance between the heads of the two vectors. That is, it is the distance between the points \((1,0)\) and \(\left(\cos\left(\frac{\pi}{4} + \frac{\pi}{32} t\right), \sin\left(\frac{\pi}{4} + \frac{\pi}{32} t\right)\right). \) Then

\[ \|\mathbf{v}_1 - \mathbf{v}_2\| = \sqrt{\left(1 - \cos\left(\frac{\pi}{4} + \frac{\pi}{32} t\right)\right)^2 + \sin^2\left(\frac{\pi}{4} + \frac{\pi}{32} t\right)} = \sqrt{2 - 2\cos\left(\frac{\pi}{4} + \frac{\pi}{32} t\right)}. \]

\[ \frac{d}{dt}\|\mathbf{v}_1 - \mathbf{v}_2\| = \frac{2\sin\left(\frac{\pi}{4} + \frac{\pi}{32} t\right)\cdot \frac{\pi}{32}}{2\sqrt{2 - 2\cos\left(\frac{\pi}{4} + \frac{\pi}{32} t\right)}}. \] When \( t = 8, \) \( \frac{d}{dt}\|\mathbf{v}_1 - \mathbf{v}_2\| = \frac{\pi \sqrt{2}}{64}. \) B

13. I is a \( 3 \times 3 \) matrix. II is a \( 2 \times 2 \) matrix. III is a \( 3 \times 3 \) matrix. IV is a \( 2 \times 2 \) matrix. D

14. The area of \( R \) is given by

\[ \int_{0}^{1} (2-x^2-x^3) \, dx = 2x - x^3 - x^4 \bigg|_{0}^{1} = \frac{17}{12}. \] The transformation matrix is

\[ T = \begin{bmatrix} 5 & -2 \\ 1 & 2 \end{bmatrix} \] and \( \det T = 12. \) The area of \( R' \) is the area of \( R \) multiplied by the determinant of the transformation matrix. \( \frac{17}{12} \cdot 12 = 17. \) C

15. The first equation implies \( x = 6 - z. \) Plugging into the second equation gives \( y + z = 6. \) This combined with the first equation implies \( x = y \) and the vector \( \mathbf{v} \) will be of the form \( \mathbf{v} = (t, t, 6 - t). \)

\[ \|\mathbf{v}\| = \sqrt{3t^2 - 12t + 36}. \] We can minimize \( 3t^2 - 12t + 36 \) for simplicity. \( 6t - 12 = 0 \) so \( t = 2 \) and hence \( \|\mathbf{v}\| = \sqrt{4 + 4 + 16} = 2\sqrt{6}. \) B

16. Notice that \( y = x^4 + 3 \) and \( x \in [1, \sqrt{2}]. \)

\[ \int_{1}^{\sqrt{2}} (x^4 + 3) \, dx = \left[ \frac{x^5}{5} + 3x \right]_{1}^{\sqrt{2}} = \frac{19\sqrt{2} - 16}{5}. \] E

17. We want to find a vector \((x, y, z)\) such that \( x + z = 0 \) and \(-x + 2y + z = 0. \) Choose \( x = 1. \) Then \( z = -1 \) and \( y = 1. \) \((1,1,-1)\) is normal to both vectors. D

18. \(|A_n| = \sqrt{n^2 + 2n - \sqrt{n^2 + 2n + 1}} = -\frac{1}{\sqrt{n^2 + 2n + \sqrt{n^2 + 2n + 1}}}. \lim_{n \to \infty} |A_n| = 0. \) A
19. The area of the triangle is the absolute value of \[ \left| \begin{array}{c} 0 \\ t/2 \\ 2 \end{array} \right| = \left| 4t - t^2 \right| \times 1 = \left| 4t - \frac{5}{4}t^2 \right|. \] Since this does not change signs when \( t \in [0,3] \), we need only maximize \( 4t - \frac{5}{4}t^2 \). \( 4 - \frac{5}{2}t = 0 \) so \( t = \frac{8}{5} \). The area is then \( 4 \cdot \frac{8}{5} - \frac{5}{4} \left( \frac{8}{5} \right)^2 = \frac{16}{5} = 3.2 \). Checking the end points, the area would be 0 and .75, so the maximum is 3.2. C

20. Notice that, in general, the \( b_{i,j} \) element of \( B^T B \) is the dot product of the \( i \)th column of \( B \) and the \( j \)th column of \( B \). Since \( A^T A = I \), it follows that the magnitude of each column is 1 and that the columns are all orthogonal. Therefore I and II are both true. A counterexample for III is the matrix \( \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \) whose determinant is \(-1\). A counterexample for IV is \( \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \). B

21. \( \begin{vmatrix} 1 & 0 & 5 \\ 0 & 4 & 1 \\ -2 & 3 & 2 \end{vmatrix} = 1 \cdot 5 - 0 \cdot 5 \cdot 8 = 45 \). C

22. \( f'(x) = (2 - \sqrt{3})e^{(2 - \sqrt{3}x)} \). \( f'(0) = 2 - \sqrt{3} \). Two points of the tangent line are the origin and \( (1,2 - \sqrt{3}) \). If we rotate these two points \( 30^\circ \), they will determine the rotated line. The rotation matrix is \( \begin{bmatrix} \cos(30^\circ) & -\sin(30^\circ) \\ \sin(30^\circ) & \cos(30^\circ) \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} \). The origin will clearly be mapped to itself. The slope of the rotated line is 1. B

23. From the last equation, \( 5x - 2 = y + z \) so \( x + y + z = 6x - 2 \). Using Cramer’s Rule, \[ x = \begin{vmatrix} -1 & 3 & 1 \\ 13 & 2 & 3 \\ 2 & -1 & -1 \end{vmatrix} = -1,1,3,-19,1,-17 = 39 \] \( x + y + z = 4 \). C

24. The speed is given by \( \|r'(0)\| \). \( r'(t) = (6t + 2)i + (\cos(t) - t \sin(t))j + e^k \). \( r'(0) = 2i + j + k \) and \( \|r'(0)\| = \sqrt{6} \). D
25. Since \( \det A = \frac{1}{6} \), \( \lim_{n \to \infty} \sum_{i=1}^{n} \det \left( A^i \right) = \lim_{n \to \infty} \sum_{i=1}^{n} \left( \det A \right)^i = \sum_{i=1}^{\infty} \left( \frac{1}{6} \right)^i = \frac{1}{1 - \frac{1}{6}} = \frac{1}{5} \). \( \text{B} \)

26. \( M \) is row equivalent to the identity matrix, hence the rank of \( A \) is 4. \( \text{E} \)

27. It is easy to see that \( A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \). Using a Maclaurin expansion,

\[
\cos(A) = I - \frac{A^2}{2!} + \frac{A^4}{4!} - \frac{A^6}{6!} + \text{L} = \begin{bmatrix} 1 - \frac{1}{2!} + \frac{1}{4!} - \text{L} & 0 - \frac{2}{2!} + \frac{4}{4!} - \text{L} \\ 0 & 1 - \frac{1}{2!} + \frac{1}{4!} - \text{L} \end{bmatrix} = \begin{bmatrix} \cos(1) & -\sin(1) \\ 0 & \cos(1) \end{bmatrix} \text{. A}
\]

28. \[
\begin{vmatrix} 2 - 3a & 3a + 1 \\ a - 2 & 1 - a \end{vmatrix} = (2 - 3a)(1 - a) - (3a + 1)(a - 2) = 3a^2 - 5a + 2 - (3a^2 - 5a - 2) = 4 \text{. B}
\]

29. \( (-1, 3) \oplus (5, 0) = \| (4, 3) \| - [-5 + 0] = 5 + 5 = 10 \text{. D} \)

30. \[
\frac{2 \cdot 1 - 1 \cdot 2 + 2 \cdot 6 + 3}{\sqrt{2^2 + (-1)^2 + 2^2}} = \frac{15}{3} = 5 \text{. C}
\]