1) \[2007 = 3^2 \cdot 17 \cdot 19, \quad 1596 = 2^2 \cdot 3 \cdot 7 \cdot 19.\] The greatest common divisor (gcd) is \[\text{gcd} = 3\cdot19 = 57\] C

2) There are only three triplets \((A, B, C)\) that work: \((1, 9, 0), (5, 9, 2),\) and \((6, 9, 3).\) A

3) 9 is always divisible by \(999\ldots9999\) which is always one less than \(10^n\) which is always one less than \(10^n\) B

4) D

5) I, II, and IV are NOT prime. D

6) \[10^3 < 32^2 < 33^2 < 34^2 < 35^2 < 36^2 < 11^3.\] 32 + 33 + 34 + 35 + 36 = 170 B

7) \[2007 \equiv 7 \pmod{100}, \quad 2007 \equiv 7 \pmod{100} \equiv 7^3 \pmod{100} \equiv 43 \pmod{100} C\]

8) For \((n-1)!\) to be divisible by \(n,\) it is equivalent to \(n!\) divisible by \(n^2.\) This means that \(n\) cannot be prime. The only composite number that this doesn’t hold true for is \(n = 4.\) There are 10 prime integers between 1 and 30 inclusive, thus there are 11 for which it is not valid. C

9) The only possible positive integers that has \(d(n) = 3\) are the squares of prime numbers. The ones less than 1000 would be: \(2^2, 3^2, 5^2, 7^2, 11^2, 13^2, 17^2, 19^2, 23^2, 29^2, 31^2\) A

10) In order for \(n!\) to not be congruent to \(0 \pmod{200},\) then it cannot contain all of the factors of 200. \(10! = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7\) which is congruent to \(0 \pmod{200}\) and \(9! = 2^7 \cdot 3^4 \cdot 5 \cdot 7\) which is not congruent. Thus the largest value is \(9\) E

11) \(1! \equiv 1 \pmod{10}, \quad 2! \equiv 4 \pmod{10}, \quad 3! \equiv 7 \pmod{10}, \quad 4! \equiv 6 \pmod{10}, \quad 5! \equiv 5 \pmod{10}, \quad 6! \equiv 6 \pmod{10}, \quad 7! \equiv 7 \pmod{10} \equiv 3 \pmod{10}, \quad 8! \equiv 8^4 \pmod{10} \equiv 6 \pmod{10}, \quad 9! \equiv 9^4 \pmod{10} \equiv 3 \pmod{10}, \quad 10! \equiv 1 \pmod{10} \pmod{10} \equiv 3 \pmod{10}\) C

12) The number of zeros in \(2007!\) is found by dividing 2007 by powers of 5: \[\left\lfloor \frac{2007}{5} \right\rfloor = 401, \quad \left\lfloor \frac{2007}{25} \right\rfloor = 80, \quad \left\lfloor \frac{2007}{125} \right\rfloor = 16, \quad \left\lfloor \frac{2007}{625} \right\rfloor = 3.\] 401 + 80 + 16 + 3 = 500.

When raising a number to a power, we multiply the power by the number of zeros to get the total number of zeros: \(500 \cdot 2 = 1000\) D

13) A theorem states that if we’re given an order \(q\) of an element and the group of integers under multiplication \((\mod p)\) where \(p\) is prime, then \(q \mid (p-1).\) Since \(3 \nmid 10,\) then there cannot be any elements of order 3. A

14) Any number with more than 3 digits will reduce the number of digits. Any number with 1 digit will increase the number of digits. Any number with 2 digits will give a number of 2 or 3 digits. When a cycle (or limit) occurs, all of the numbers must have the following property: must be 2 or 3 digits long, sum of hundreds and tens digits must be equal to units digit, and the units digit must be either 2 or 3. (This is because all numbers will lead to numbers of this form, and these numbers are closed.) We only have to consider the following numbers now: 22, 33, 112, 202, 123, 213, and 303. Noting that \(33 \rightarrow 22 \rightarrow 202 \rightarrow 303 \rightarrow 123, 213 \rightarrow 123, 112 \rightarrow 123,\) and \(123 \rightarrow 123\) proves that every number has a limit of 123. D

15) The Fibonacci numbers less than 2007 are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 577, 610, 987, 1597. The ones that satisfy \(F_n^2 + 1\) being divisible by ten must have a 3 or a 7 in the one's digit: 3, 13, 233, 377, 987, 1597. D

16) We solve this similar to a base-6 problem. \(2007 \pmod{26} = 5, \quad (2007 - 5)/26 = 77\)

\[77 \pmod{26} = 25, \quad (77 - 25)/26 = 2.\] \(2 \Rightarrow b, \quad 25 \Rightarrow y, \quad 5 \Rightarrow e\) C

17) The largest odd square less than 2007 is \(43^2 = 1849.\) 2(22)−1 = 43
1^2 + 3^2 + 5^2 + ... = \sum_{n=1}^{\infty} (2n-1)^2 = \sum_{n=1}^{\infty} 4n^2 - 4n + 1 = \frac{4(22)(23)(45)}{6} - \frac{4(22)(23)}{2} + 22(1) = 14190 \quad \text{A}

18) 723_8 = 7(64) + 2(8) + 3 = 467_{10}, 124_5 = 25 + 2(5) + 4 = 39_{10}, 723_8 + 124_5 = 506_{10}, 506_{10} = 622_9 \quad \text{B}

19) Since 102! = 102 \cdot 101 \cdot 100!, the greatest common divisor is 100! \quad \text{E}

20) Simplifying the expression yields \( m! = \frac{20n!}{n!-20} \). For \( n \) larger than 6, the ratio \( \frac{20n!}{n!-20} \) will not be an integer and \( n \) cannot be smaller than 4. For \( n = 4 \) we have \( m! = 120 \Rightarrow m = 5 \). For \( n = 5 \) we have \( m! = 120 \Rightarrow m = 4 \). Since we can switch \( m \) and \( n \) we have two solutions. \quad \text{C}

21) In order to have an exact number of cents, our cost must be a multiple of $0.25. Let's denote this as 25n where \( n \) is the number of quarters we need. In order to have an exact number of dollars, when we multiply our cost without tax to 1.04, we should get an integer: 1.04(25n) = 26n, which must be divisible by 100; or 13n must be divisible by 50. This yields \( n = 50 \), so our pre-tax cost is 25(50) = 1250 cents = $12.50. \quad \text{A}

22) I is TRUE since all primes greater than 3 are odd and \( a \equiv -1 \pmod{p} \) would correspond to an even integer. II is TRUE from \( p \) being odd and is trivially true for \( p = 2 \). III is TRUE by Wilson's Theorem. IV is TRUE from Fermat's Little Theorem. ALL are true. \quad \text{A}

23) 7056 = 2^4 \cdot 3^2 \cdot 7^2. \quad \text{Thus the smallest value for } n \text{ is } 2^2 \cdot 3 \cdot 7 = 84 \quad \text{C}

24) 4155 = 5 \cdot 6! + 4 \cdot 5! + 3 \cdot 4! + 0 \cdot 3! + 1 \cdot 2! + 1 \cdot 1! \quad 543011 \quad \text{D}

25) The pattern is the following: 15, 28, 39, 48, 55, 60, 63, 64, 72, 72, 75, 76, 78, 78, 78 \quad \text{B}

26) When first eliminating all the integers with a 1, there will be \( 9^3 = 729 \) numbers remaining. If he was to eliminate all integers with a 2 from the remaining, there would be \( 8^3 = 512 \) numbers remaining. However, he keeps all numbers with BOTH a 2 and a 3. The possibilities are: X23, X32, 2X3, 3X2, 23X, and 32X, where X is any digit from 0-9, excluding 1. This means there are 54 possibilities, but we have double counted the following numbers: 223, 232, 322, 332, 323, and 233. Subtracting these gives 48 numbers with BOTH a 2 and a 3 from the remaining. Adding these back in gives \( 512 + 48 = 560 \). This means he crossed out \( 1000 - 560 = 440 \) \quad \text{A}

27) Trying the first few: \( a_1 = 1, a_2 = 3, a_3 = 12, a_4 = 60, ... \). The terms are related to the factorial sequence by division of 2 and shifted to the left by 1. Thus \( a_n = \frac{(n+1)!}{2} \) \quad \text{C}

28) 2007 minutes is equivalent to 33 hours and 27 minutes. If we move forward by exactly 24 hours, the time remains the same. By moving 4 hours forward, we reach 0:07. Thus we must move 5 hours and 27 minutes forward from 0:07, which corresponds to 5:34 \quad \text{B}

29) \( 28x \equiv 2 \pmod{54} \) is the same thing as saying: find an integer solution to \( 28x + 54y = 2 \), a linear Diophantine equation. This is equivalent to \( 14x + 27y = 1 \). Using the Euclidian algorithm, an initial solution is \( x_0 = 2 \) and \( y_0 = -1 \). All possible solutions are in the form \( x = 2 + 27t \) and \( y = -1 - 14t \). The integer values of \( x \) less than 100 are 2, 29, 56, and 83. \quad \text{D}

30) 56 = \( 2^3 \cdot 7 \); 72 = \( 2^3 \cdot 3^2 \); The LCM is \( 2^3 \cdot 3^2 \cdot 7 = 504 \) \quad \text{C}