PART I: Mr. Jensen's Math Class

1. **ANS: 6 (C)**. We have that $(x-8)(x+5)=10(x-1)$. Simplifying gives $x^2-13x-30=(x+2)(x-15)=0$, which gives $A=15$, so the sum of digits is 6.

2. **ANS: 4 (A)**. Firstly, note that $\sin\theta\cos\theta\cos 2\theta\cos 4\theta=\frac{1}{2}\sin 2\theta\cos 2\theta\cos 4\theta=\frac{1}{4}\sin 4\theta\cos 4\theta=\frac{1}{8}\sin 8\theta$, which evaluated at the given θ is $\frac{1}{16}$. So, the answer is $B=64e^{\ln\frac{1}{16}}=4$.

3. **ANS: 3 (C)**. We can rewrite this equation as $2\tan^2\theta=\sec\theta+1$. Since $\sec^2\theta=\tan^2\theta+1$, we have that $2\sec^2\theta-\sec\theta-3=(2\sec\theta-3)(\sec\theta+1)=0$. This means that $\sec\theta=\frac{3}{2},-1$. For $\sec\theta=\frac{3}{2}$, we have two solutions that sum to 2π. For $\sec\theta=-1$, we just have $\theta=\pi$. So, the sum of solutions is 3π, meaning that $C=3$.

4. **ANS: 3 (A)**. We can simplify the equation: $(x-4)^2+4(y+1)^2=16$, or $\frac{(x-4)^2}{16}+\frac{(y+1)^2}{4}=1$. This is an ellipse, with center $(4,-1)$. The point with maximal y-coordinate is thus $(4,1)$, and we know that $c^2=12$, so the length of the latus rectum is $\frac{2b^2}{a}=2$. Thus, the coordinate of the rightmost focus is $(4+2\sqrt{3},-1)$, so the endpoint with maximal y-coordinate is $(4+2\sqrt{3},0)$. The line passing through these two points is $x+2y\sqrt{3}=4+2\sqrt{3}$. So, we have that $D=|1+n+p|=3$.

5. **ANS: 15 (D)**. The equation can be factored as $(2x-1)(x^2+3x+5)(x+2)(x^2-x+3)$, so the product of the non-rational roots is $E=15$.~~ion,~~

6. **ANS: 9 (B)**. The probability that we land on A is $\frac{15}{40}=\frac{3}{8}$. The probability that we land on E is also $\frac{3}{8}$. So, the probability that we land on A and E in some order is $2\cdot\frac{3}{8}\cdot\frac{3}{8}=\frac{9}{32}$.

PART 2: All's Fair in Love and War

7. **ANS: 149 (D)**. Let $z=a+bi$, where a,b are integers. We must have that $a^2+b^2\leq 49$. We then have the following pairs for points purely in the first quadrant: $(1-6,1),(1-6,2),(1-6,3),(1-5,4),(1-4,5),(1-3,6)$. This gives us $6+6+6+5+4+3=30$. So, there are 120 in the four quadrants, plus 29 on the axes, for a total of 149.

8. **ANS: ¼ (C)**. We know that $\sin\theta=\frac{8}{17}$, so we must have that $\cos\theta=-\frac{15}{17}$, since θ is in the second quadrant. So, we have that $\cot\frac{\theta}{2}=\frac{\sin\theta}{1-\cos}=\frac{8}{32}=\frac{1}{4}$.

9. **ANS: -77/36 (A)**. From previously, we know that $\tan\theta=-\frac{8}{15}$. Now, consider τ. Since secant is negative, it's in the second quadrant. Thus, $\tan\tau=-\frac{3}{4}$. So, we compute:

$$\tan(\tau+\theta)=\frac{\left(-\frac{3}{4}-\frac{8}{15}\right)}{1-\frac{3}{4}\cdot\frac{8}{15}}=-\frac{45+32}{60-24}=-\frac{77}{36}$$

10. **ANS: -25 (A)**. Catherine now wants to make sure that Daniel's love for her isn't imaginary, so she tests his complex number skills and asks him to rotate the point $(-4,5)$ by $-390°$ counterclockwise. What coordinates should Daniel end up with (assuming he calculates correctly)? We can do this on the Argand plane. We compute $(-4+5i)cis(-390°)=(-4+5i)cis(330°)$. Then, we know that $cis(330°)=\frac{\sqrt{3}}{2}-\frac{1}{2}i$. So, $(-4+5i)\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)=-2\sqrt{3}+\frac{5}{2}+2i+\frac{5\sqrt{3}}{2}i$.

11. **ANS: 460 (D)**. We have that $4-4i\sqrt{3}=8cis(300°)$. So, $w=\sqrt{2}cis(50°),\sqrt{2}cis(110°),\sqrt{2}cis(170°),\sqrt{2}cis(230°),\sqrt{2}cis(290°),\sqrt{2}cis(350°)$. The only one in the second quadrant is $\sqrt{2}cis(230°)$, giving us an answer of $230\sqrt{2}$.

12. **ANS: 56/81 (C)**. Daniel has passed all of Catherine's tests! As a final assessment, Catherine has a set of complex numbers of the form $a+bi$ for integers $a,b\in[-1,1]$, from which Daniel will choose two, z and w, randomly and without replacement. Catherine will go on a date with Daniel if $\frac{z}{w}$ is also in the set. What is the probability Catherine will go on a date with Daniel?

Just for demonstration, the elements of the set are the following:

$$-1,0,1,-1-i,-i,1-i,-1+i,i,1+i$$

We can visualize this with a table, with w on the vertical and z on the horizontal:

	-1	0	1	-1-i	-i	1-i	-1+i	i	1+i
-1	x	x	x	x	x	x	x	x	x
0									
1	x	x	x	x	x	x	x	x	x
-1-i		x		x		x	x		x
-i	x	x	x	x	x	x	x	x	x
1-i		x		x		x	x		x
-1+i		x		x		x	x		x
i	x	x	x	x	x	x	x	x	x
1+i		x		x		x	x		x

So, our final probability is $\frac{9+9+5+9+5+5+9+5}{81}=\frac{56}{81}$.

PART 3: Shenanigans in Space

13. **ANS: $(\sqrt{34}-3)/5$ (B)**. We know that $c = 5$ and that $\frac{2b^2}{a} = 12$, where $c^2 = a^2 - b^2 = 25$. So, $b^2 = 6a$, meaning that $a^2 - 6a - 25 = 0$. Solving, we obtain $a = 3 + \sqrt{34}$. The eccentricity is thus $\frac{c}{a} = \frac{5}{\sqrt{34}+3} = \frac{5(\sqrt{34}-3)}{25} = \frac{\sqrt{34}-3}{5}$.

14. **ANS: 18 (B)**. The initial vertical velocity is $24\sin 30° = 12$ meters per second. We can model the ball's movement with the equation $-\frac{4}{2}x^2 + 12x + 0$. We can rewrite this as $-2(x-3)^2 + 18$, which means that the vertex is $(3, 18)$, so the maximum height is 18.

15. **ANS: $2\pi\sqrt{3}$ (B)**. Note that $y = r\sin\theta$, so we can rewrite this as $r = \frac{3}{2-\frac{y}{r}} = \frac{3r}{2r-y}$, or $2r = y + 3$. Squaring both sides, we obtain $4r^2 = (y+3)^2$. But since $r^2 = x^2 + y^2$, we see that $4x^2 + 4y^2 = y^2 + 6y + 9$. Consolidating, $4x^2 + 3y^2 - 6y + 3 = 4x^2 + 3(y-1)^2 = 12$. Then, we have that the area is $2\pi\sqrt{3}$.

16. **ANS: 2/3 (D)**. Using the double angle formula, we can rewrite the inequality as $2\sin^2\theta + 5\sin\theta - 3 > 0$, or $(2\sin\theta - 1)(\sin\theta + 3) > 0$. This inequality is true when $\sin\theta > \frac{1}{2}$, or when $30° < \theta < 150°$. So, the probability is $\frac{2}{3}$.

17. **ANS: 18π (C)**. Note that the cow travels forward and back $\frac{\pi}{2}$ to center position, then backwards and forward π to center position, then $\frac{3\pi}{2}$, and so on. Sixty degrees represents a length of $\frac{1}{6} \cdot 18\pi = 3\pi$ from center position, so we need to calculate $2 \cdot \frac{\pi}{2} + 2 \cdot \pi + \cdots + 2 \cdot \frac{5\pi}{2} + 3\pi = 18\pi$.

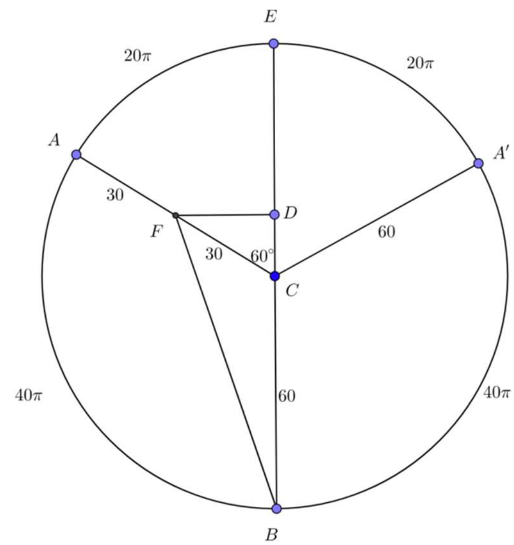
18. **ANS: $\sqrt{5}/5$ (C)**. We can rewrite the equation of the parabola as $(y-3)^2 = 8(x-1)$. The vertex is $(1,3)$, and the focus is thus $(3,3)$, which is where Mr. Lu is. The endpoints of the latus rectum are $(3,7)$ and $(3,-1)$. The minimum distance is just from the vertex to the focus, which is 2. The maximum distance is from the vertex to an endpoint, which by the Pythagorean theorem is $\sqrt{2^2+4^2} = 2\sqrt{5}$. So, the ratio is $\frac{\sqrt{5}}{5}$.

PART 4: Party Preparations

19. **ANS: $30\sqrt{7}$ (C)**. The cone can be "unraveled" by cutting along segment AC to form a sector with arc ABA'. Since arc AB is half the circumference of the base of the cone, it has length 40π, which is also the length of arc BA'. Since the slant height is 60, $AC = BC = A'C = 60$ as well. Thus, arc BE has length 60π, which means that arcs AE and EC both have length 20π. So, angle AEC has measure $60°$ and ACB has measure $120°$. The Giffenbug is at the midpoint of AC, which is point F in the figure below. So, we need to find the straight-line distance between F and B, which can be done using law of cosines on triangle FCB:

$$BF^2 = CF^2 + CB^2 - 2(CF)(CB)\cos 120° = 6300$$

So, $BF = 30\sqrt{7} = Y$.



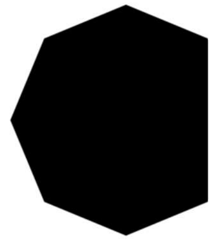
20. **ANS: $288\sqrt{3}/25$ (C)**. The base of the current tetrahedron is an equilateral triangle with side length 66 and has volume $\frac{(6)(6)\sqrt{3}}{4} = 9\sqrt{3}$, and the current height of the triangle can be found using $V = \frac{Bh}{3}$, or $\frac{36\sqrt{3}}{5} = \frac{1}{3} \cdot 9\sqrt{3} \cdot h$, so the current height is $\frac{12}{5}$. Now, let's take a look at the right triangle formed by the height of the tetrahedron as a leg and the altitude from D to edge AB as the hypotenuse. The hypotenuse has length 4, since each half of face ABD is a (3, 4, 5) right triangle. The length of one of the legs is $\frac{12}{5}$, so this right triangle is also a (3, 4, 5) triangle. Let θ be the angle between face ABD and the base. So, $\sin\theta = \frac{\frac{12}{5}}{4} = \frac{3}{5}$, which means that doubling the angle gives $\sin 2\theta = 2 \cdot \frac{3}{5} \cdot \frac{4}{5} = \frac{24}{25}$. Then, the new height becomes $4\sin 2\theta = \frac{96}{25}$. Therefore, the new volume is $\frac{1}{3} \cdot 9\sqrt{3} \cdot \frac{96}{25} = \frac{288\sqrt{3}}{25}$.

21. **ANS: 16π (B)**. We'll just calculate the volume for each axis. For the x-axis, we have a cone of radius 4 and height 3, with volume $\frac{\pi}{3}(4)(4)(3) = 16\pi$. For the y-axis, we have a cone of radius 3 and height 4, with volume $\frac{\pi}{3}(3)(3)(4) = 12\pi$. The greatest of the two volumes is 16π.

22. **ANS: 95 (B)**. For Stackelberg's graphs, since the coefficient of θ is even, we must multiply each $2k$ by 2 to find the number of petals, which gives us $4 + 8 + 12 + 16 + 20 = 60$ petals. For Bertrand's graphs, since the coefficient of θ is odd, we don't multiply by anything, so we just have $3 + 5 + 7 + 9 + 11 = 35$ petals, for a total of 95 petals.

23. **ANS: $(3\sqrt{2} + 1)/2$ (C)**. Note that $\frac{x^8-1}{x-1} = x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0$, so the roots to this equation are the 8th roots of unity except for 1. On the Argand plane, the

eighth roots of unity form an octagon. However, since we only have seven of the eight points, we end up with a figure like the one below:



The distance from the original center to each vertex is 1. So, we can find the area of this shape by finding the area of each triangle formed by segments from the center to the vertices. This gives us $6 \cdot \frac{1}{2} \cdot 1 \cdot 1 \cdot \sin 45° + \frac{1}{2} \cdot 1 \cdot 1 \cdot \sin 90° = 3 \cdot \frac{\sqrt{2}}{2} + \frac{1}{2} = \frac{3\sqrt{2}+1}{2}$.

24. **ANS: 17 (E)**. Multiply the numerator and denominator by $2 - xi$ to obtain $\frac{36(2-xi)}{4+x^2} = \frac{72}{4+x^2} - \frac{36}{4+x^2}i$. We must require that $4 + x^2$ divide 72. We can have $4 + x^2 = 4, 6, 8, 9, 12, 18, 24, 36, 72$, which gives 17 solutions for x.

PART 5: Casino Party

25. **ANS: 16/455 (E)**. Let's just consider the 16 Kings, Queens, Jacks, and Aces, since the order of the other cards doesn't matter. The probability we want is $6 \cdot \frac{4}{16} \cdot \frac{4}{15} \cdot \frac{4}{14} \cdot \frac{4}{13} = \frac{16}{455}$, since the King, Queen, and Jack can be reordered in $3! = 6$ ways.

26. **ANS: 0.72 (C)**. Given the result of the first game, Mr. Frazer can win the second and third, or he can lose the second and win the third. The probability of the former happening is $(0.8)(0.8) = 0.64$. The probability of the latter happening is $(0.2)(0.4) = 0.08$. This gives us a total probability of 0.72.

27. **ANS: 8/11 (A)**. The number of full house hands is $13\binom{4}{3}12\binom{4}{2} = 13(12)(24) = X$. The number of flush hands is $4\binom{13}{5} = Y$. We then obtain $\frac{(13)(12)(24)}{(13)(12)(11)(3)} = \frac{8}{11}$.

28. **ANS: 17/32 (C)**. We'll casework by the largest card picked, noting that the total number of possibilities is 64.

    a. There is exactly one triplet whose largest card is 1: (1, 1, 1).
    b. For triplets whose largest card is 2, we can have (1, 2, 2), of which there are three, and (2, 2, 2), of which there is only 1, giving us a total of 4.
    c. For triplets whose largest card is 3, we can have (1, 3, 3), (2, 2, 3), (2, 3, 3), and (3, 3, 3), giving a total of $3 + 3 + 3 + 1 = 10$.
    d. For triplets whose largest card is 4, we can have (1, 4, 4), (2, 3, 4), (2, 4, 4), (3, 3, 4), (3, 4, 4), and (4, 4, 4), giving a total of $3 + 6 + 3 + 3 + 3 + 1 = 19$.

So, we have a total of $1+4+10+19=34$, or a probability of $\frac{34}{64}=\frac{17}{32}$.

29. **ANS: 1/3 (B)**. We are conditioning on the fact that there is a goat behind door 2. So, if all three doors contain goats, with probability $\frac{4}{9}$, you will not win the car. If there are goats behind doors 1 and 2, and a car behind door 3, with probability $\frac{2}{9}$, you win the car with probability $\frac{1}{2}$. If there are goats behind doors 2 and 3, and a car behind door 1, with probability $\frac{2}{9}$, you win the car with probability $\frac{1}{2}$. Finally, if there are cars behind doors 1 and 3, and a goat behind door 2, with probability $\frac{1}{9}$, you win the car with probability 1. This gives a total probability of

$$\frac{4}{9}\cdot 0+\frac{2}{9}\cdot\frac{1}{2}+\frac{2}{9}\cdot\frac{1}{2}+\frac{1}{9}\cdot 1=\frac{1}{3}$$

30. **ANS: 251 (C)**. We know that $1011!=1\cdot 2\cdot\ldots\cdot 1010\cdot 1011$. Zeros appear at the end of a number when a 2 is multiplied with a 5. So, we just need to count the number of 2's and 5's in this number's prime factorization. However, note that there is an abundance of 2's relative to 5's, so we'll just count the number of 5's. There are $\left\lfloor\frac{1011}{5}\right\rfloor=202$ multiples of 5, $\left\lfloor\frac{1011}{25}\right\rfloor=40$ multiples of 25, $\left\lfloor\frac{1011}{125}\right\rfloor=8$ multiples of 125, and $\left\lfloor\frac{1011}{625}\right\rfloor=1$ multiple of 625. So, when we count the total number of 5's, we compute the sum $202+40+8+1=251$ zeros. Note that for the number 625, each of its four 5's are counted once in the numbers 202, 40, 8, and 1.