1. B
2. C
3. B
4. A
5. C
6. D
7. A
8. B
9. D
10. E
11. A
12. B
13. C
14. D
15. A
16. C
17. B
18. C
19. C
20. B
21. D
22. A
23. D
24. B
25. B
26. E
27. C
28. A
29. D
30. A

1) There are 12 face cards, 13 clubs, and 3 face cards that are clubs in the deck. Thus, $\frac{52-(12+13-3)}{52} = \frac{15}{26}$. $A = 15$ and $B = 26$, so $A + B = 41$.

2) $10! + 100 = 3628900$. The sum of the digits of this is 28.

3) The probability that neither ball is blue is $\frac{6}{10} \cdot \frac{5}{9} = \frac{1}{3}$. The complement of this is $\frac{2}{3}$.

4) The constant term is the sum of the coefficients of the $x^0 2^5 \left(-\frac{1}{x^2}\right)^0$ $x^2 2^2 \left(-\frac{1}{x^2}\right)^1$ terms. This is equal to $\frac{5!}{0!5!0!}(2^5) - \frac{5!}{2!2!1!}(2^2) = 32 - 120 = -88$.

5) $1000 = 2^3 5^3$. $320 = 2^6 5$. The GCF of these is $2^3 5 = 40$. 1000 has 16 factors and 40 has 8 factors, so $\frac{1}{2}$ of the factors of 1000 are also factors of 320.

6) $p(\alpha \cap \beta) = p(\alpha) + p(\beta) - p(\alpha \cup \beta) = 0.2$. $p(\alpha|\beta) = \frac{p(\alpha\cap\beta)}{p(\beta)} = \frac{0.2}{0.6} = \frac{1}{3}$.

7) There are $\frac{6!}{3!} = 120$ ways to arrange the letters of ARMADA. $\frac{5!}{2!} = 60$ of them start with an A. Out of the words starting DA, $3! = 6$ of them start with DAA. Counting fro m here shows that DAMARA is in position 68.

8) Stars-and-bars can be used to show that there are $\binom{5+3-1}{5} = 21$ possible ways for the dice to add to 8. The ones that include a 4 are (1,3,4) and (2,2,4), which accounting for orderings makes 9 ways that include a 4. $\frac{9}{21} = \frac{3}{7}$.

9) The set of possible n is the set of numbers with factorization $2^a 5^b$ for whole numbers a and b. The reciprocal of the squares of one of these numbers is $\frac{1}{4^a 25^b}$

$$\sum_{b=0}^{\infty}\sum_{a=0}^{\infty}\frac{1}{4^a 25^b} = \sum_{b=0}^{\infty}\frac{1}{4^a}\sum_{a=0}^{\infty}\frac{1}{25^b} = \frac{4}{3}\cdot\frac{25}{24} = \frac{25}{18}$$

10) MISSISSIPPI contains 1 M, 4 I's, 4 S's, and 2 P's. To be a palindrome, a permutation must have the M in the middle and 2 I's, 2 S's, and 1 P on either side. $\frac{5!}{2!2!1!} = 30$.

11) For this question (as well as Question 12), it is helpful to make a chart of the probabilities of each minion dying. Let A be the minion with 2 health, and let B be the minion with 3 health. Listing them in the chart represents them taking 1 damage. The second column is the reciprocal of the probability that the minions take damage in this order.

$AABB$	4
$ABAB$	8
$ABBA$	16
$BAAB$	8
$BABA$	16
$BBAA$	16
$ABBB$	16
$BABB$	16
$BBAB$	16
$BBBA$	8

The 3rd, 5th, and 6th rows represent the minion with 2 hp dying on the fourth hit of Minivolcano. $\frac{1}{16}+\frac{1}{16}+\frac{1}{16}=\frac{3}{16}$.

12) The 1st through 6th rows represent the minion with 2hp dying. $\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{8}+\frac{1}{16}+\frac{1}{16}=\frac{11}{16}$.

13) Now let A be the minion with 2 health, and let B be any minion that is not A. A similar chart as before can be constructed, taking into account the random selection of minions.

$AABB$	16
$ABAB$	16
$ABBA$	16
$BAAB$	12
$BABA$	12
$BBAA$	8
$ABBB$	16
$BABB$	12
$BBAB$	8
$BBBA$	4

The 1st through 6th rows represent the minion with 2hp dying. $\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{12}+\frac{1}{12}+\frac{1}{8}=\frac{23}{48}$.

14) There are 21 vertical and 20 horizontal lines that can hold the sides of the rectangle, which has 2 vertical and 20 horizontal lines. The number of possible rectangles is $\binom{21}{2}\binom{20}{2}$.

15) $\binom{2n}{n}=\frac{(2n)!}{(n!)^2}$. This is equivalent to the following:

$$\frac{\sqrt{4\pi n}\left(\frac{2n}{e}\right)^{2n}}{2\pi n\left(\frac{n}{e}\right)^{2n}}=\frac{2\sqrt{\pi n}\cdot 4^n\left(\frac{n}{e}\right)^{2n}}{2\pi n\left(\frac{n}{e}\right)^{2n}}=\frac{4^n}{\sqrt{\pi n}}$$

$A=4$ and $B=4$, so $A+B=8$.

16) $36=2^2 3^2$, so it has 9 factors. If $\alpha=36$, then β can be any of α's 8 proper factors. The remaining pairs can be constructed by making one of α and β divisible by 4 and the other divisible by 9: (18,4), (18,12), (12,9), and (9,4). This is 12 pairs.

17) The probability of picking a boy from Connor's class is $\frac{1}{2}\cdot\frac{12}{20}=\frac{3}{10}$. The probability of picking a boy from Arnav's class is $\frac{1}{2}\cdot\frac{9}{25}=\frac{9}{50}$. $\frac{\frac{9}{50}}{\frac{3}{10}+\frac{9}{50}}=\frac{3}{8}$.

18) Let the probability the coin lands on heads be p. Then he probability Jackson wins is the probability he flips heads, plus the probability he flips heads after Couper and he both flip tails, plus the probability he flips heads after Couper and he both flip tails twice, and so on. This equals $p+p(1-p)^2+p(1-p)^4+\cdots=\frac{p}{1-(1-p)^2}=\frac{1}{2-p}$. Setting this equal to $\frac{5}{8}$ yields $p=\frac{2}{5}$.

19) This is a standard application of the Pigeonhole Principle.

20) This is a polynomial of degree 18 with all positive coefficients, meaning it has 19 terms.

21) 101 is prime. $\sum_{n=1}^{\infty}\left\lfloor\frac{101^3}{101^n}\right\rfloor=101^2+101+1$. Thus, $\kappa=101^2+101+1$.

22) In sum notation:

$$\sum_{n=1}^{12} n + \sum_{n=13}^{19} 2n + \sum_{n=20}^{20} 5n = 100 + \sum_{n=1}^{19} 2n - \sum_{n=1}^{12} n = 100 + 19 \cdot 20 - \frac{12 \cdot 13}{2} = 402$$

The expected amount of damage is $\frac{402}{20} = 20.1$.

23) Paloma can choose the order of the 4 flowers at the top, as well as which of the 2 directions she wants to start drawing on for each. When she finishes with the flowers, she can choose the order of the 2 petals, as well as which of the 2 directions she wants to start drawing on for each. $4! \cdot 2^4 \cdot 2! \cdot 2^2 = 3072$.

24) $\cos x + \cos 2x = 2\cos^2 x + \cos x - 1 = (2\cos x - 1)(\cos x + 1)$. This is only greater than 0 when the two factors are both positive (since $(\cos x + 1)$ cannot be negative), or when $x \in \left[0, \frac{\pi}{3}\right] \cup \left[\frac{5\pi}{3}, 2\pi\right]$. $\frac{\pi/3}{2\pi} = \frac{1}{6}$.

25) The discriminant of $x^2 + ax + a + \frac{5}{4}$ is $a^2 - 4\left(a + \frac{5}{4}\right) = a^2 - 4a - 5 = (a-5)(a+1)$. For the polynomial to have distinct real roots, either $a > 5$ or $a < -1$. 14 out of the 21 possible values of a satisfy this condition, so the probability is $\frac{2}{3}$.

26) Translating the problem to geometric probability, consider the square with side length 2 in the first quadrant that is tangent to both coordinate axes. The probability that Steven and Melissa meet is the proportion of the square that is contained between the lines $y = x - 0.5$ and $y = x + 1$ (Steven and Melissa meet when $y = x$), which can be found by subtracting the areas of two isosceles right triangles from the square. $\frac{4 - \frac{1}{2} - \frac{9}{8}}{4} = \frac{19}{32}$.

27) A right triangle is formed by connecting each end of the diameter of a circle to another point on the circle. Since all regular polygons are cyclical, a diameter can be selected (300 possibilities), leaving 598 points for the third point of the triangle left. This means there are $300 \cdot 598$ possible right triangles triangles out of $\binom{600}{3}$ total triangles. $\frac{300 \cdot 598}{\binom{600}{3}} = \frac{3}{599}$.

28) This problem is analogous to asking the number of Dyck words of length 7, which is well known to be the 7th Catalan number, where $C_n = \frac{1}{n+1}\binom{2n}{n}$. Plugging in $n = 7$ yields $C_7 = 429$.

29) The number 9999999 has a sum of digits of 63. A seven-digit number with sum of digits of 59 has 1 subtracted from some combination of 4 digits in 9999999; by stars-and-bars, there are $\binom{4+7-1}{4} = 210$ possible numbers. To be divisible by 11, the sum of the 1st, 3rd, 5th, and 7th digits must differ from the sum of the 2nd, 4th, and 6th digits by a multiple of 11. Because of the restriction on the total sum of the digits, this difference must be exactly 11. Thus, the sum of the odd-indexed digits must be 35, and the sum of the even-indexed digits must be 24.

Trivially, the odd-indexed digits are some permutation of {8,9,9,9}, leading to 4 possible orderings. The even-indexed digits can be {6,9,9} (3 orderings), {7,8,9} (6 orderings), or {8,8,8} (1 ordering). Thus, there are $4 \cdot 10 = 40$ possible seven-digit numbers, meaning the probability is $\frac{4}{21}$. $A = 4$ and $B = 21$, so $A + B = 25$.

30) $\alpha_n + i\beta_n = \alpha_{n+1}\alpha_{n-1} - \beta_{n+1}\beta_{n-1} + i\beta_{n+1}\alpha_{n-1} + i\alpha_{n+1}\beta_{n-1} = (\alpha_{n+1} + i\beta_{n+1})(\alpha_{n-1} + i\beta_{n-1})$. Let $c_n = \alpha_n + i\beta_n$. Note that $\alpha_n^2 + \beta_n^2 = |c_n|^2$. $c_n = c_{n+1}c_{n-1}$, so $c_0 = 2 + 7i$ and $c_1 = 1 + 8i$. $c_n = c_{n+1}c_{n-1}$, so since $c_{n+1} = c_{n+2}c_n$, $c_n = c_{n+2}c_nc_{n-1}$ or $c_{n-1} = \frac{1}{c_{n+2}}$. Shifting the indices, we have $c_{n+3} = \frac{1}{c_n}$, which means $c_{n+6} = c_n$. Note that $2019 \equiv 3 \bmod 6$. Thus, $|c_{2019}|^2 = \frac{1}{|c_0|^2} = \frac{1}{53}$. $P = 1$ and $Q = 53$, so $P + Q = 54$.