1. B
2. A
3. D
4. D
5. A
6. B
7. B
8. D
9. C
10. D
11. B
12. B
13. D
14. B
15. D
16. E
17. B
18. E
19. E
20. B
21. C
22. B
23. B
24. C
25. C
26. A
27. A
28. D
29. C
30. C

1. Notice that $(1+i)^2 = 2i$ and $(1-i)^2 = -2i$. Then $(1+i)^{20}+(1-i)^{18} = (2i)^{10}+(-2i)^9 = -1024-512i$ $\boxed{B}$

2. $\frac{3-2i}{1+i} + \frac{2+i}{4+5i} = \frac{(3-2i)(1-i)}{(1+i)(1-i)} + \frac{(2+i)(4-5i)}{(4+5i)(4-5i)} = \frac{1-5i}{2} + \frac{13-6i}{41} = \frac{41(1-5i)+2(13-6i)}{82} = \frac{67-217i}{82}$ $\boxed{A}$

3. Since absolute value is multiplicative, $|(3-4i)(5+12i)(24+7i)| = (5)(13)(25) = 1625$ $\boxed{D}$

4. $(1+\sqrt{3}i)^{10} = (2\operatorname{cis}(\frac{\pi}{3}))^{10} = 2^{10}\operatorname{cis}(\frac{10\pi}{3}) = 1024\operatorname{cis}(\frac{4\pi}{3}) = 1024(-\frac{1}{2}-\frac{\sqrt{3}}{2}) = -512-512\sqrt{3}$ $\boxed{D}$

5. $(3+5i)\operatorname{cis}(\frac{\pi}{4}) = (3+5i)(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}) = -\sqrt{2}+4i\sqrt{2}$ $\boxed{A}$

6.
$$7z_1 = 2(2z_1+z_2)+3z_1-2z_2 = 2(5+4i)+4-i = 14+7i \to z_1 = 2+i$$
$$z_2 = 5+4i-2(2+i) = 1+2i$$
$$|z_1+z_2| = |3+3i| = 3\sqrt{2}$$
$\boxed{B}$

7. $\frac{12}{3-i\sqrt{3}} = \frac{12(3+i\sqrt{3}}{(3-i\sqrt{3})(3+i\sqrt{3})} = \frac{12(3+i\sqrt{3})}{12} = 3+i\sqrt{3} = 2\sqrt{3}\operatorname{cis}(\frac{\pi}{6})$ $\boxed{B}$

8. By listing out a couple of terms we see that:
$$a_0 = i$$
$$a_1 = -1+i$$
$$a_2 = -1-i$$
$$a_3 = -1+i$$
Clearly the terms repeat every 2. Following the pattern we know that $a_{100} = -1-i$ $\boxed{D}$

9. Since $f(x)$ has real coefficients, then $6+2i$ must also be a root of $f(x)$. Then $f(x) = (x-(6-2i))(x-(6+2i))$ and $f(8) = (8-(6-2i))(8-(6+2i)) = (2+2i)(2-2i) = 8$ $\boxed{C}$

10. Let $z = a+bi$, then $z\overline{z} = (a+bi)(a-bi) = a^2+b^2 = 1$. Thus the graph is a circle $\boxed{D}$

11. What is a_{2018} if $a_0 = a_1 = i$ and $a_n = a_{n-1}a_{n-2}$ for $n \geq 2$?

    We list out terms of the sequence until it repeats. We know the sequence will repeat mod4 since the exponents follow the Fibonacci sequence.

$$a_0 = i$$
$$a_1 = i$$
$$a_2 = -1$$
$$a_3 = -i$$
$$a_4 = i$$
$$a_5 = 1$$
$$a_6 = i$$
$$a_7 = i$$

    Thus we know that the sequence repeats every 6 terms. Then since $2018 \equiv 2 \bmod 6$, $a_{2018} = a_2 = -1$ $\boxed{B}$

12. $f(x)$ will not intersect the x-axis if the discriminant is less than 0. Then $D = n^2 - 4n < 0$ which is true for $0 < n < 4$. Thus there are 3 values of n $\boxed{B}$

13. Let $z = a+bi$, then $z\overline{z} = (a+bi)(a-bi) = a^2+b^2$. We can find possible answers by examining mod 4. Since the quadratic residues of 4 are 0,1, a^2+b^2 can only be 0,1, or 2 mod 4. Then 2019 is clearly impossible $\boxed{D}$

14. Letting $z = a+bi$ we get the equation $a^2 - b^2 + 2abi = -8+6i$. By setting the real and imaginary parts of the equation equal to each other we obtain, $a^2 - b^2 = -8$ and $2ab = 6$. Substituting $b = \frac{3}{a}$ gives $a^2 - (\frac{3}{a})^2 = -8$. Multiplying by a^2 and rearranging gives $a^4 + 8a^2 - 9 = 0$. Factoring gives the solutions $(a^2+9)(a^2-1) = 0 \Rightarrow a = \pm 3i, \pm 1$. Finally this gives us our solutions of $z = 1+3i, -1-3i$. Then $|z-\overline{z}| + |z-\overline{z}| = 6+2 = 8$ $\boxed{B}$

15. We can complete the 4th power which gives $4x^4 - 16x^3 + 24x^2 - 16x + 4 + 13 = 4(x-1)^4 + 9 = 0$. Solving for x we get

$$4(x-1)^4 + 9 = 0$$
$$(x-1)^4 = -\frac{9}{4}$$
$$x = \frac{\sqrt{6}}{2}\operatorname{cis}\left(\frac{\pi}{4} + \frac{k\pi}{2}\right) + 1$$

The roots clearly form a square with diagonal length of $\sqrt{6}$. Thus the area is $\frac{\sqrt{6}\sqrt{6}}{2} = 3$ $\boxed{D}$

16. Let x be the value that the expression converges to. Then $x = \sqrt{3 + (i2\sqrt{3})x}$.

$$x = \sqrt{3 + (i2\sqrt{3})x}$$
$$x^2 = 3 + x2i\sqrt{3}$$
$$x^2 - x(2i\sqrt{3}) - 3 = 0$$
$$x = \frac{2i\sqrt{3} \pm \sqrt{-12+12}}{2}$$
$$x = i\sqrt{3}$$

$\boxed{E}$

17. Expressing $\sqrt{i}$ in exponential form we get $\sqrt{i} = e^{\frac{\pi i}{2}^{\frac{1}{2}}} = e^{\frac{\pi i}{4}}$. Then $(\sqrt{i})^i = e^{\frac{\pi i}{4}^{i}} = e^{\frac{-\pi}{4}}$ $\boxed{B}$

18. $\prod_{n=1}^{360}(\operatorname{cis}(n))^{360-n} = \prod_{n=1}^{360}(\operatorname{cis}(360n - n^2)) = \prod_{n=1}^{360}(\operatorname{cis}(-n^2)) = \operatorname{cis}(-\sum_{n=1}^{360} n^2) = \operatorname{cis}(-\frac{(360)(361)(721)}{6}) = \operatorname{cis}(-(60)(361)(721)) = \operatorname{cis}(-(60)(1)(1)) = \frac{1}{2} - \frac{i\sqrt{3}}{2}$ $\boxed{E}$

19. Let $d = gcd(k, 360)$. Then notice that $a_0 r^{\frac{360}{d}} = a_0 \operatorname{cis}\left(\frac{360k}{d}\right)$. Since $d|gcd(k,360) \rightarrow d|k$, clearly $360|\frac{360k}{d} = 360\frac{k}{d}$. Moreover, $\frac{360}{d} \leq 360$, where equality is achieved when $d = 1$. Thus, in order to satisfy our condition we must have $gcd(k, 360) = 1$. The number of integers that satisfy this condition is $\phi(360) = 96$ $\boxed{E}$

20. Notice that on every 4th move Mr. Lu is at $-\frac{n}{2} - \frac{ni}{2}$. Then after his 2016th move he is at $-1008 - 1008i$. After his 2017th move he will be at $1009 - 1008i$, and finally after his 2018th move he will be at $1009 + 1010i$ $\boxed{B}$

21. Note that the 5th roots of unity are $1, \operatorname{cis} 72°, \operatorname{cis} 144°, \operatorname{cis} 216°, \operatorname{cis} 288°$. The sum of roots is $1 + \operatorname{cis} 72° + \operatorname{cis} 144° + \operatorname{cis} 216° + \operatorname{cis} 288° = 0$, which implies that the sum of the real parts of the roots must also be 0. This means that $1 + \cos 72° + \cos 144° + \cos 216° + \cos 288° = 0$. Then since cos is an even function we know that $\cos 216° = \cos(-216°) = \cos 144°$ and $\cos 288° = \cos(-288°) = \cos 72°$. Then, $1 + \cos 72° + \cos 144° + \cos 216° + \cos 288° = 1 + 2(\cos 72° + \cos 144°) = 0$. Thus, $\cos 72° + \cos 144° = -\frac{1}{2}$ $\boxed{C}$

22.

$$(\cos(\theta) + i\sin(\theta))^4 = \text{cis}^4(\theta)$$
$$\cos^4(\theta) + 4i\cos^3(\theta)\sin(\theta) - 6\cos^2(\theta)\sin^2(\theta) - 4i\cos(\theta)\sin^3(\theta) + \sin^4(\theta) = \text{cis}(4\theta)$$

We set the real parts equal to each other which gives

$$\begin{aligned}\cos(4\theta) &= \cos^4(\theta) - 6\cos^2(\theta)\sin^2(\theta) + \sin^4(\theta)\\ &= \cos^4(\theta) - 6\cos^2(\theta)(1-\cos^2(\theta)) + (1-\cos^2(\theta))^2\\ &= \cos^4(\theta) - 6\cos^2(\theta) + 6\cos^4(\theta)) + (1 - 2\cos^2(\theta) + \cos^4(\theta))\\ &= 8\cos^4(\theta) - 8\cos^2(\theta) + 1\end{aligned}$$

$\boxed{B}$

23. Graphing $2017 - 2018i$ we see that it has an angle that is very slightly less than $\frac{7\pi}{4}$. Then by DeMoivre's Theorem, $(2017-2018i)^{50}$ will have an angle that is slightly less than $\frac{350\pi}{4} = \frac{175\pi}{2} = \frac{3\pi}{2}$. Therefore it is in the third quadrant $\boxed{B}$

24. What is the distance between the foci of the conic $13x^2 + 10xy + 13y^2 = 72$?

The angle of rotation needed to eliminate the xy term on a conic is given by the formula $\cot(2\theta) = \frac{A-C}{B}$. Then $\cot(2\theta) = \frac{13-13}{10} = 0$, so the needed angle is $\frac{\pi}{4}$. Now consider any point (x_0, y_0) on the original conic, using cis we can rotate it by $\frac{\pi}{4}$.

$$\begin{aligned}(x_0+iy_0)(\text{cis}(\tfrac{\pi}{4})) &= (x_1+iy_1)\\ (x_0+iy_0) &= (x_1+iy_1)(\text{cis}(-\tfrac{\pi}{4}))\\ (x_0+iy_0) &= (x_1+iy_1)(\tfrac{\sqrt{2}}{2} - i\tfrac{\sqrt{2}}{2})\\ (x_0+iy_0) &= \tfrac{x_1\sqrt{2}}{2} + \tfrac{y_1\sqrt{2}}{2} + i(\tfrac{y_1\sqrt{2}}{2} - \tfrac{x_1\sqrt{2}}{2})\end{aligned}$$

Then substituting $\frac{x_1\sqrt{2}}{2} + \frac{y_1\sqrt{2}}{2}$ for x_0 and $\frac{y_1\sqrt{2}}{2} - \frac{x_1\sqrt{2}}{2}$ for y_0 gives

$$\begin{aligned}13x_0^2 + 10x_0y_0 + 13y_0^2 &= 72\\ 13(\tfrac{x_1\sqrt{2}}{2} + \tfrac{y_1\sqrt{2}}{2})^2 + 10(\tfrac{x_1\sqrt{2}}{2} + \tfrac{y_1\sqrt{2}}{2})(\tfrac{y_1\sqrt{2}}{2} - \tfrac{x_1\sqrt{2}}{2}) + 13(\tfrac{y_1\sqrt{2}}{2} - \tfrac{x_1\sqrt{2}}{2})^2 &= 72\\ 8x_1^2 + 18y_1^2 &= 72\\ \tfrac{x_1^2}{9} + \tfrac{y_1^2}{4} &= 1\end{aligned}$$

Then the distance between the foci is $2c = 2\sqrt{a^2-b^2} = 2\sqrt{5}$ $\boxed{C}$

25. $\sum_{n=1}^{\infty}|P_n - P_{n-1}| = \sum_{n=1}^{\infty}|(\frac{1}{2}+\frac{1}{2}i)P_{n-1} - P_{n-1}| = \sum_{n=1}^{\infty}|P_{n-1}||(\frac{1}{2}-\frac{1}{2}i)| = \frac{\sqrt{2}}{2}\sum_{n=1}^{\infty}|P_{n-1}| = \frac{\sqrt{2}}{2}\sum_{n=1}^{\infty}|P_0(\frac{1}{2}+\frac{1}{2}i)^{n-1}| = \frac{\sqrt{2}}{2}\sum_{n=1}^{\infty}|13+84i||(\frac{1}{2}+\frac{1}{2}i)^{n-1}| = \frac{85\sqrt{2}}{2}\sum_{n=1}^{\infty}|(\frac{1}{2}+\frac{1}{2}i)^{n-1}| = \frac{85\sqrt{2}}{2}\sum_{n=1}^{\infty}(\frac{\sqrt{2}}{2})^{n-1} = \frac{85\sqrt{2}}{2}\frac{1}{1-\frac{\sqrt{2}}{2}} = \frac{85\sqrt{2}}{2-\sqrt{2}} = 85\sqrt{2}+85$ $\boxed{C}$

26. By the Binomial expansion theorem, the coefficient is $\binom{8}{4}(i)^4(2)^4 = 1120$ $\boxed{A}$

27.

$$\begin{aligned} |18+ni| &= k \\ \sqrt{324+n^2} &= k \\ 324+n^2 &= k^2 \\ k^2-n^2 &= 324 \\ (k-n)(k+n) &= 324 \end{aligned}$$

Let d be a divisor of 324. If we have $k-n=d$ and $k+n=\frac{324}{d}$, then $n=\frac{\frac{324}{d}-d}{2}$. This implies that both $\frac{324}{d}$ and d must be even (it is impossible for both to be odd). Then the number of possible values for n is equal to the number of factors of 81, which is 5. However, d and $\frac{324}{d}$ are symmetric, which means we must correct for overcounting. Then there are $\frac{5+1}{2}=3$ possible values of n. In addition, the case where $d=18$ gives $n=0$ which is not a valid solution. Finally we conclude that there are only 2 solutions, namely 80 and 24 $\boxed{A}$

28. Let $z = r\,\text{cis}(\theta)$, then $2z^2 = 2r^2\,\text{cis}(2\theta)$ and $3z^3 = -3r^3\,\text{cis}(3\theta) = 3r^3\,\text{cis}(3\theta+\pi)$. By Law of Cosines,

$$\begin{aligned} (P_1P_2)^2 &= r^2+4r^4-4r^3\cos(\theta) \\ (P_2P_3)^2 &= 4r^4+9r^6-12r^5\cos(\pi+\theta) \\ &= 4r^4+9r^6+12r^5\cos(\theta) \end{aligned}$$

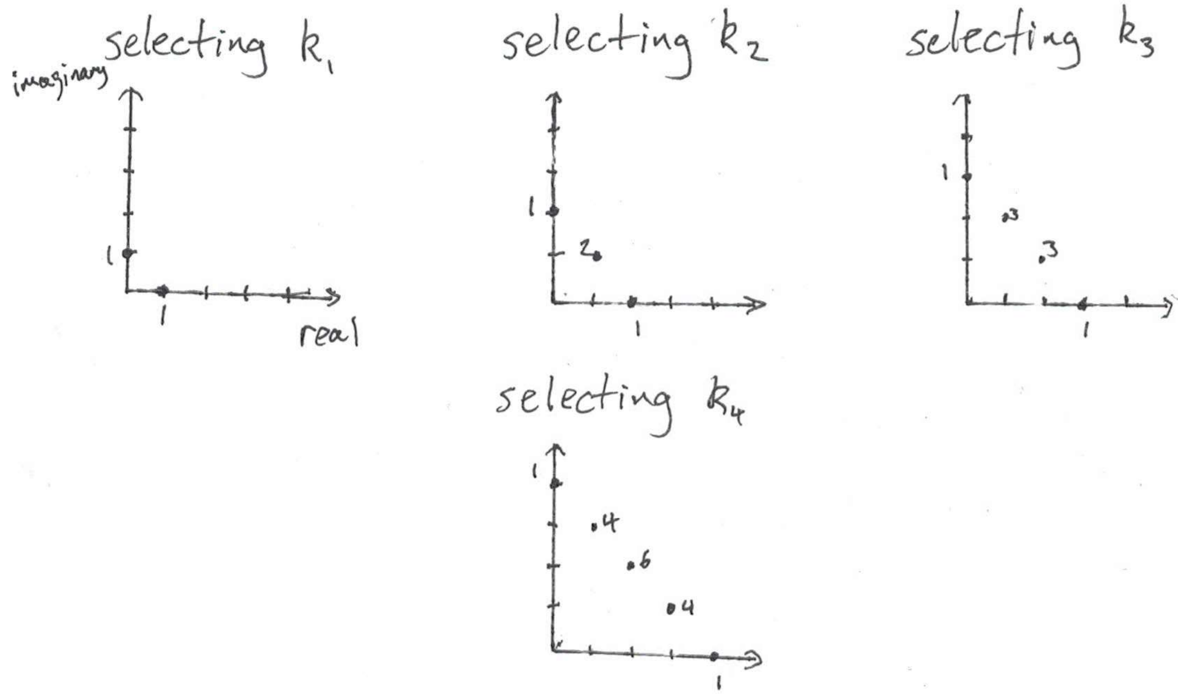
Then we must have

$$\begin{aligned} r^2+4r^4-4r^3\cos(\theta) &= 4r^4+9r^6+12r^5\cos(\theta) \\ 1+4r^2-4r\cos(\theta) &= 4r^2+9r^4+12r^3\cos(\theta) \\ 9r^4-1 &= 12r^3\cos(\theta)+4r\cos(\theta) \\ (1-3r^2)(1+3r^2) &= 4r\cos(\theta)(3r^2+1) \\ (1-3r^2) &= 4r\cos(\theta) \\ 1-3(x^2+y^2) &= 4x \\ -3(x^2+y^2)-4x &= -1 \\ x^2+\frac{4}{3}x+y^2 &= \frac{1}{3} \\ (x+\frac{2}{3})^2+y^2 &= \frac{7}{9} \end{aligned}$$

Then the area enclosed by all such z is $\frac{7\pi}{9}$. $\lceil\frac{7\pi}{9}\rceil = 3$ $\boxed{D}$

29. Plugging in $a, b = 0$ we see that $f^2(0) = f(0)$. This implies that $f(0) = 0, 1$. However, it is obvious that $f(0) \neq 0$ since $f(a) = f(a)f(0) = 0$ which violates the property that one-to-one. Thus $f(0) = 1$.
If we plug in $a = x$ and $b = -x$, we find that $f(x)f(-x) = f(0) = 1 \rightarrow f(x) = \frac{1}{f(-x)}$.
In addition, $-1 = f(-1) = f(-\frac{2}{3})f(-\frac{1}{3})$. So our sum reduces to $f(\frac{2}{3}) + f(-\frac{2}{3}) = \frac{1}{f(-\frac{2}{3})} + f(-\frac{2}{3}) = -f(-\frac{1}{3}) - \frac{1}{f(-\frac{1}{3})} = -\frac{f^2(-\frac{1}{3})+1}{f(-\frac{1}{3})}$.
Finally, $-1 = f(-1) = f(-\frac{2}{3})f(-\frac{1}{3}) = f(-\frac{1}{3})f(-\frac{1}{3})f(-\frac{1}{3}) = f^3(-\frac{1}{3}) \rightarrow f^3(-\frac{1}{3}) + 1 = 0$.
Obviously $f(-\frac{1}{3}) \neq -1$ since $f(-1) = -1$, which leaves us with $f^2(-\frac{1}{3}) - f(-\frac{1}{3}) + 1 = 0 \rightarrow f^2(-\frac{1}{3}) + 1 = -f(-\frac{1}{3})$.
Then our final sum is $-\frac{f^2(-\frac{1}{3})+1}{f(-\frac{1}{3})} = -\frac{f(-\frac{1}{3})}{f(-\frac{1}{3})} = -1$
Alternatively, it is easy to see that either $f(x) = \text{cis}(\pi x)$ or $f(x) = \text{cis}(-\pi x)$ satisfies the properties given. Then $\text{cis}(\frac{2\pi}{3}) + \text{cis}(-\frac{2\pi}{3}) = -1$ $\boxed{C}$

30. We count the number of ways to reach each possibility each turn. The number of ways to reach a point is the sum of the number of ways to reach the point from the points adjacent below (selecting 1) and to the left (selecting 0).



Now we can break down cases by magnitude. There are 3 possible magnitudes after the selection of k_4, namely, 4, $\sqrt{10}$,$\sqrt{8}$. There are 2 total ways to reach a magnitude of 4, 8 total ways to reach a magnitude of $\sqrt{10}$, and 6 total ways to reach a magnitude of $\sqrt{8}$. Clearly there are $2 \cdot 2 + 8 \cdot 8 + 6 \cdot 6 = 104$ ways for Ben and David to match magnitudes out of a total of $16 \cdot 16$ possibilities. Then the probability that Ben wins is $\frac{104}{256} = \frac{13}{32}$
$\boxed{C}$