FLORIDA ASSOCIATION OF MU ALPHA THETA
2019 NATIONAL CONVENTION: ALPHA MATRICES & VECTORS
SOLUTIONS

1. B $\begin{cases} 2x+3y=8 \\ x-2y=-3 \end{cases}$ translates to the matrix equation $\begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ -3 \end{bmatrix}$.

2. A $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 1(-1)+2(3) & 1(2)+2(-4) \\ 3(-1)+4(3) & 3(2)+4(-4) \end{bmatrix} = \begin{bmatrix} 5 & -6 \\ 9 & -10 \end{bmatrix}$.

3. C $\begin{bmatrix} -4 & 5 \\ 2 & -2 \end{bmatrix}^{-1} = \frac{1}{(-4)(-2)-2(5)}\begin{bmatrix} -2 & -5 \\ -2 & -4 \end{bmatrix} = \frac{1}{-2}\begin{bmatrix} -2 & -5 \\ -2 & -4 \end{bmatrix} = \begin{bmatrix} 1 & \frac{5}{2} \\ 1 & 2 \end{bmatrix}$

4. C We know $\det(rA) = r^n\det(A)$ for some scalar r and n is the number of rows of A. Also, $\det(A^T) = \det(A)$ and $\det(A^{-1}) = 1/\det(A)$. Since the determinant is a multiplicative operator, $\det(B) = k^n(k)(k)(1/k) = k^{n+1}$.

5. E One way to check for singularity is to see if any row or column is a linear combination of the other rows or columns. In choice A, rows 1 and 2 are opposites so it is singular. In choice B, columns 1 and 3 are multiples of one another so it is singular. In choice C, rows 1 and 2 add to row 3, so this matrix is singular. In choice D, a row of zeros makes the matrix singular. Thus, no matrix is non-singular.

6. E For $\begin{cases} 2x+ky=4 \\ -3x-2y=-6 \end{cases}$ to be a dependent system, the coefficients of the two rows must be scalar multiples. Since the second equation is -3/2 times the first, we require $k(-3/2) = -2$ and $k = 4/3$. However, when $k = 4/3$, the two equations are simply multiples of each other. Therefore, the system is still consistent.

7. C Note $A^2 = \begin{bmatrix} -11 & -15 \\ 9 & -14 \end{bmatrix}$. Thus, $\begin{bmatrix} -11 & -15 \\ 9 & -14 \end{bmatrix} + \alpha\begin{bmatrix} 2 & -5 \\ 3 & 1 \end{bmatrix} + \beta\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ so $\begin{bmatrix} -11+2\alpha+\beta & -15-5\alpha \\ 9+3\alpha & -14+\alpha+\beta \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Thus, $\alpha = -3, \beta = 17$ and $\alpha + \beta = 14$. Note that the existence of such a quadratic is guaranteed by the Cayley-Hamilton Theorem.

8. B Rather than repeated multiplication, note that $\frac{A}{2}$ represents a rotation counterclockwise by 30 degrees about (0,0). So $\left(\frac{A}{2}\right)^7$ is a rotation counterclockwise by 210 degrees. $A^7 = 128\begin{bmatrix} -\frac{\sqrt{3}}{2} & 1 \\ -1 & -\frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} -64\sqrt{3} & 64 \\ -64 & -64\sqrt{3} \end{bmatrix}$

9. D Choice D satisfies these four conditions of RREF:
1. A row full of zeros must occur below a row with at least one nonzero entry.
2. The leftmost nonzero entry of a row is 1. This is called a pivot.
3. A pivot of a row is the only nonzero entry of its column.
4. For two pivots, one in row i, column j and the other in row s, column t, if $i > j$ then $s > t$.

10. B The 2 vectors using $(-2,1,3)$ as the origin are $< 3,4,1 >$ and $< 7,1,2 >$. The area of the triangle is half the magnitude of their cross product $< 7,1,-25 >$. The area is $\frac{\sqrt{675}}{2} = \frac{15\sqrt{3}}{2}$.

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11. B $\begin{vmatrix} -3 & x+3 \\ 2-x & 2 \end{vmatrix} = -3(2) - (2-x)(x+3) = 8$ so $-6 + x^2 + 3x - 2x - 6 = 8$ and $x^2 + x - 20 = 0$ so $(x + 5)(x - 4) = 0$ and $x = -5$ or $x = 4$. The sum of these values is -1.

12. C There are (2)(2)(2)(2) = 16 possible matrices. To be invertible, the determinant $ad - bc$ must be nonzero, so it can be either 1 or -1. If $ad - bc = 1$, then $ad = 1$ and $bc = 0$. So $a = d = 1$ and at least one of b, c is 0 resulting in 3 such cases.
If $ad - bc = -1$, then $ad = 0$ and $bc = 1$. This similarly results in 3 such cases.
Thus, there are 6 out of 16 ways for a probability of 3/8.

13. D $2A + \begin{bmatrix} -1 & 0 \\ -1 & 3 \\ 2 & -4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -3 & -1 \\ 8 & -4 \end{bmatrix} \rightarrow 2A = \begin{bmatrix} 4 & 4 \\ -2 & -4 \\ 6 & 0 \end{bmatrix} \rightarrow A = \begin{bmatrix} 2 & 2 \\ -1 & -2 \\ 3 & 0 \end{bmatrix}$

14. C I is not true for all matrices, for example, $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ but $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$. II (the associative property of multiplication) and III (the distributive property) are true.

15. D Since $x = \frac{\begin{vmatrix} 2 & 3 \\ -1 & 5 \end{vmatrix}}{\begin{vmatrix} 4 & 3 \\ 1 & 5 \end{vmatrix}}$, we know the coefficient matrix is $\begin{bmatrix} 4 & 3 \\ 1 & 5 \end{bmatrix}$. Also from $\begin{vmatrix} 2 & 3 \\ -1 & 5 \end{vmatrix}$ we know the constant matrix is $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Hence, the system is $\begin{bmatrix} 4 & 3 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ or $\begin{cases} 4x + 3y = 2 \\ x + 5y = -1 \end{cases}$.

16. C The magnitude of $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is $\sqrt{x^2 + y^2 + z^2}$. To be an integer, we see that $x^2 + y^2 + z^2$ must be a perfect square. Of the choices, we see that the magnitude of $\begin{pmatrix} 1 \\ 4 \\ -8 \end{pmatrix}$ is $\sqrt{(1)^2 + (4)^2 + (-8)^2} = \sqrt{81} = 9$.

17. A The magnitude of $\begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}$ is $\sqrt{(-1)^2 + (2)^2 + (-2)^2} = 3$. To be in the opposite direction, we scale $\vec{v}$ by $-\frac{1}{3}$ to get $\begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$.

18. C For $\begin{bmatrix} -1 & x & -3 \\ x & 0 & 0 \\ 3 & -2 & x+1 \end{bmatrix}$ to not be invertible, the determinant must be 0:

$-1(0) - x(x(x+1) — (2)(-3)) + 3(0) = -x(x^2 + x - 6) = 0$
Factoring, $-x(x + 3)(x - 2) = 0$ so $x = 0, -3, 2$ whose sum is -1.

19. D We need $\vec{u} \cdot \vec{v} = 2x + 6 = 0$ so $x = -3$ and $\vec{u} \cdot \vec{w} = -2 - 6y = 0$ so $y = -1/3$. The product $xy = -3(-1/3) = 1$.

20. C $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ so A is not idempotent.

$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 0 \\ -1 & 1 & 1 \end{bmatrix}^2 = \begin{bmatrix} 2 & 1 & 2 \\ 2 & 0 & 2 \\ 4 & 1 & 2 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 0 \\ -1 & 1 & 1 \end{bmatrix}$ so B is not idempotent.

$\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}^2 = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ so C is idempotent.

21. B From above, we know that powers of choice A will either be the identity or A.

$\begin{bmatrix} 5 & -3 & 2 \\ 15 & -9 & 6 \\ 10 & -6 & 4 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so B is nilpotent.

22. C One way to find the angle between two vectors is to consider the dot product. $\vec{u} \cdot \vec{v} = 3(2) - 1(1) = 5$, but also $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos(\theta) = \sqrt{10}\sqrt{5}\cos(\theta)$. Thus, $\cos(\theta) = \frac{5}{\sqrt{50}} = \frac{1}{\sqrt{2}}$. Thus, the acute angle between the vectors is $45°$.

23. C We need to solve the system $\begin{cases} -a + 3b - 2c = 0 \\ 4a + 2b + c = 0 \end{cases}$. Multiplying the first row by 4 and adding it to the second, we find that $2b = c$. Then $a = 3b - 2c = 3b - 2(2b) = -b$. So $(a, b, c) = (-b, b, 2b)$ for some real b. We see that $a = -1$, $b = 1$, $c = 2$ fits this description. So the line is $-x + y = 2$, which is $\sqrt{2}$ away from the origin.

24. C Consider the 4x4 matrix $A = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix}$. We know that $a + f + k + p = 3$

Now $5A = \begin{bmatrix} 5a & 5b & 5c & 5d \\ 5e & 5f & 5g & 5h \\ 5i & 5j & 5k & 5l \\ 5m & 5n & 5o & 5p \end{bmatrix}$ whose trace is $5(a + f + k + p) = 5(3) = 15$.

25. A $m^2 = |\vec{v} + \vec{w}|^2 = (\vec{v} + \vec{w}) \cdot (\vec{v} + \vec{w}) = |\vec{v}|^2 + 2\vec{v} \cdot \vec{w} + |\vec{w}|^2$. Also $n^2 = |\vec{v} - \vec{w}|^2 = (\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) = |\vec{v}|^2 - 2\vec{v} \cdot \vec{w} + |\vec{w}|^2$. Thus, $m^2 - n^2 = 4\vec{v} \cdot \vec{w}$ and hence $\vec{v} \cdot \vec{w} = \frac{1}{4}(m^2 - n^2)$.

26. C We want the two vectors to not be a scalar multiple of one another. The only vector that does this is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

27. C For $A = \begin{bmatrix} -2 & 3 & 7 \\ x & 5 & z \\ y & -2 & -1 \end{bmatrix}$ to be symmetric, $A = \begin{bmatrix} -2 & 3 & 7 \\ 3 & 5 & -2 \\ 7 & -2 & -1 \end{bmatrix}$, thus $x = 3$, $y = 7$, $z = -2$ and the sum is 8.

28. E For $\begin{cases} x+y=8 \\ x+z=11 \\ y+z=13 \end{cases}$, we can add all three equations: $2(x+y+z)=32$ so $x+y+z=16$.
Subtracting each equation to get $x=3, y=5, z=8$, and $xyz=120$.

29. E None of the answer choices is true.

30. A If $\begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = c\begin{bmatrix} x \\ y \end{bmatrix}$ then $\begin{bmatrix} 2-c & -4 \\ -1 & -1-c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. If $\begin{bmatrix} x \\ y \end{bmatrix}$ is non-zero, then $\begin{bmatrix} 2-c & -4 \\ -1 & -1-c \end{bmatrix}$ is singular, so its determinant is zero.
$(2-c)(-1-c)-(-4)(-1)=(c-2)(c+1)-4=c^2-c-6=0$.
The solutions $c=3$ and $c=-2$ have a product of -6.
Note that these values of c are called “eigenvalues”.