<u>Alpha Sequences and Series 2019 Answer Key</u>

1. B
2. A
3. C
4. C
5. C
6. D
7. A
8. D
9. A
10. A
11. C
12. D
13. B
14. C
15. A
16. C
17. D
18. B
19. B
20. D
21. C
22. C
23. A
24. D
25. C
26. B
27. C
28. D
29. A
30. A

Alpha Sequences and Series 2019 Solutions

1. B. We have $a_1 = a_3 - 2d = 30 - 2 \times 7 = 16$.
2. A. Rearrange: $(107 - 67) + (64 - 24) + (84 - 44) + (1923 - 1833) = 40 \times 4 = 160$
3. C. Note that 3 distinct positive integers sum up to at least $1 + 2 + 3 = 6$. These aren't in geometric progression, but we can get $1 + 2 + 4 = 7$ instead, and these are in geometric progression, so the answer is 7.
4. C. The key observation is that the sum of two arithmetic series is yet another arithmetic series: $(a_1 + nd_1) + (a_2 + nd_2) = (a_1 + a_2) + n(d_1 + d_2)$. Thus, the 2019^{th} term is $3 + 2019 \times (8 - 3) = 10098$
5. C. We have $3 + 9 + 27 + 81 + 243 + 729 = \frac{3^7 - 3}{2} = 1092$ by the geometric series formula.
6. D. The first observation is that the sequence $a_{a_1}, a_{a_2}, \ldots$ is itself an arithmetic sequence. To see this, note that $a_{a_k} = a_{a+(k-1)d} = a + \big((a + (k-1)d) - 1\big)d = a + ad - d + (k-1)d^2$. This is of the form $x + (k-1)y$, so it's an arithmetic progression.
   The second observation is that the sum of the first five terms of an arithmetic sequence is just five times the third term. To see this, if $b_1, b_2, \ldots$ are in arithmetic progression, then
   $$b_1 + b_2 + b_3 + b_4 + b_5$$
   $$= (b_3 - 2d) + (b_3 - d) + b_3 + (b_3 + d) + (b_3 + 2d) = 5b_3$$
   We are then looking for $5a_{a_3} = 5a_{11} = 5 \times 43 = 215$
7. A. The first term is 2 and the 30^{th} term is $2 + 29 \times (6 - 2) = 118$. The sum of the first 30 terms is then $\frac{1}{2}(2 + 118)(30) = 1800$
8. D. Suppose $m > 68$. Then there are 33 positive integers bigger than 68 but less than or equal to 100. There are only 32 such integers, so this is impossible. On the other hand, $m = 68$ is achievable with the set $\{68, 69, \ldots, 100\}$.
9. A. We have
   $$a_1 + a_2 + \cdots + a_9 + a_{10} = 10^3 + 3 \times 10 + 1 = 1031$$
   $$a_1 + a_2 + \cdots + a_9 = 9^3 + 3 \times 9 + 1 = 757$$
   So $a_{10} = 1031 - 757 = 274$
10. A. Suppose 3 is in a barbershop quartet. Note that it must be the smallest element, since 2,3,4,5 contains 4, a composite number. If 3 is the smallest number, then the largest number is of the form $3 + 3d = 3(d + 1)$, which can't be prime. The other three answer choices are contained in the two barbershop quartets (5,11,17,23) and (7,19,31,43).
11. C. The correct first term is $1 + 4d$. The first term Ellen found is $2d + 4$. Thus, $1 + 4d = 2d + 4$, so $d = 1.5$
12. D. We can manually evaluate as by noting that the coefficient of x^6 is equal to
   $$\sum_{i=0}^{6} 2^i 3^{6-i} = 3^6\left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \cdots + \left(\frac{2}{3}\right)^6\right) = \frac{3^6\left(1 - \left(\frac{2}{3}\right)^7\right)}{\frac{1}{3}}$$
   $$= 3^7 - 2^7 = 2059$$

An alternative solution, which reveals the math underlying the answer, uses the formula for an infinite geometric series as follows:

$$(1+2x+4x^2+8x^3+\dots)(1+3x+9x^2+27x^3+\dots)$$
$$=\frac{1}{1-2x}\times\frac{1}{1-3x}=\frac{3}{1-3x}-\frac{2}{1-2x}$$
$$=(3+9x+27x^2+\cdots)-(2+4x+8x^2+\cdots)$$

So the coefficient of x^n is $3^{n+1}-2^{n+1}$.

13. B. The sum of the first n odd numbers is $1+3+\cdots+(2n-1)=\frac{1}{2}\times(2n)\times n=n^2$. So the sum is just $200^2=40000$.
14. C. Let $\frac{b}{a}=\frac{3c}{b}=r$. Since the sequence is arithmetic, we have

$$b-a=c-b$$
$$1-\frac{a}{b}=\frac{c}{b}-1$$
$$1-\frac{1}{r}=\frac{r}{3}-1$$
$$3r-3=r^2-3r$$
$$r^2-6r+3=0$$
$$r=3\pm\sqrt{6}$$

We know $r>1$, so we must have $r=3+\sqrt{6}$.
15. A. Note that any string of consecutive integers is an arithmetic sequence, and that the average of the first and last terms in an arithmetic sequence is the average of all the terms. Thus, if the first slip has average 6.5, it must contain the number 1 through 12. The next strip, to average 24, must have the numbers 13 through $2\times24-13=35$. (The formula represents the fact that 13 and 35 average to 24. Continuing in this vein, the next number is $2\times98-36=160$. The final number, which is n, is $2\times232-161=303$.
16. C. For $n>2$, note that $n^2<(n+1)^2<2n^2$. In other words, the remainder when $(n+1)^2$ is divided by n^2 is $(n+1)^2-n^2$. Thus, we have

$$r_3+r_4+\cdots+r_{2019}=(4^2-3^2)+(5^2-4^2)+\cdots+(2020^2-2019^2)$$

This telescopes to 2020^2-3^2. Finally, we add $r_2=1$ to get $2020^2-3^2+1=4080392$, which leaves a remainder of 392 when divided by 1000.
17. D. We have $ar^2=7$ and $ar^6=28$. Then the eleventh term is $ar^{10}=\frac{(ar^6)(ar^6)}{ar^2}=\frac{28^2}{7}=112$
18. B. We have $\frac{\mathrm{co}}{\sin\alpha}=\frac{\sin}{\cos\alpha}$. The right-hand side is $\frac{2\sin\alpha\cos\alpha}{\cos\alpha}=2\sin\alpha$ using the double-angle formula. Then $\cos\alpha=2\sin^2\alpha=2-2\cos^2\alpha$. Solving the resulting quadratic, we get $\cos\alpha=\frac{-1+\sqrt{17}}{4}$
19. B. The sum of every four consecutive terms is 4, so the overall sum is $(4)\times25=100$.
20. D. Let the first term of the sequence be a and let the second term be b. Then we can compute the first ten terms as: $a, b, a+b, 2b+a, 3b+2a, 5b+3a, 8b+5a, 13b+8a, 21b+8a, 34b+13a$. Summing these up, we get $88b+55a=11(8b+5a)$. Therefore, the only possible value for the seventh term is just $\frac{517}{11}=47$.
21. C. Add and subtract $\sin 306°$. Then we have

$$(\sin 18^{\circ} + \sin 90^{\circ} + \sin 162^{\circ} + \sin 234^{\circ} + \sin 306^{\circ}) - \sin 306^{\circ}$$

Note that the first five angles form a regular pentagon inscribed in the unit circle. By symmetry, their sines must add up to 0, so the expression is just equal to $-\sin 306^{\circ} = \sin 54^{\circ}$

22. C. In order for the condition to hold, every third term must be equal. In particular, $Z = O = 7$ and $H = L$. Then the sum is $Z + H + O + I = 7 + O + L + I = 19$.
23. A. The fifth term of the arithmetic sequence is $1 + 4 \times 2 = 9$ and the fifth term of the geometric sequence is $1 \times 2^4 = 16$ The difference in absolute value of these two terms is $16 - 9 = 7$.
24. D. Create a new sequence $b_n = a_n - 1$. The recurrence then reduces to $b_n = b_{n-1}^2$. The product we want is then

$$S = (1 + b_0)(1 + b_0^2)(1 + b_0^4)(1 + b_0^8) \dots$$

Multiply both sides by $1 - b_0$. Then we get

$$S(1 - b_0) = (1 - b_0^2)(1 + b_0^2)(1 + b_0^4) \dots$$

$$= (1 - b_0^4)(1 + b_0^4)(1 + b_0^8) \dots$$

$$\dots$$

This product collapses indefinitely. Since $|b_0| < 1$, all terms of the form $b_0^{2^n}$ go to 0, so the right hand side converges to 1. Thus, we have

$$S = \frac{1}{1 - b_0} = \frac{2}{3}$$

25. C. Let a_3 be the third term and d be the common difference. Then the first five terms add up to $(a_3 - 2d) + (a_3 - d) + a_3 + (a_3 + d) + (a_3 + 2d) = 5a_3$. Thus, the third term is $\frac{35}{5} = 7$
26. B. Let a be the first term and r the common ratio. Then we have that

$$\frac{a + ar + ar^2}{3} = \frac{ar^3 + ar^4 + ar^5 + ar^6 + ar^7 + ar^8}{6}$$

We can factor both sides:

$$\frac{a(1 + r + r^2)}{3} = \frac{ar^3(1 + r + r^2)(1 + r^3)}{6}$$

We can cancel the a (since the sequence is nonconstant) and we can cancel $1 + r + r^2$ since that has no real roots. We're left with

$$\frac{1}{3} = \frac{r^3 + r^6}{6}$$

Solving this as a quadratic in r^3, we get $r^3 = 1, -2$. Then, the only real possible values of r are 1 and $-\sqrt[3]{2}$, which sum to $1 - \sqrt[3]{2}$.

27. C. We have $\ln(I) = 2019ln2019$ and $\ln(\ln(I)) = \ln(2019) + \ln(\ln(2019))$. At this point, it's good enough to just bound this number between 7 and 12 (you can get 7 and 12, by noting that $2 < e < 3$ and $2^{12} \approx 2019 \approx 3^7$. These bounds are crude but sufficient). Then we have $\ln(\ln(\ln(I))) \approx 2$, implying $\ln(\ln(\ln(\ln(I)))) < 1$ and $\ln(\ln(\ln(\ln(\ln(I))))) < 0$. Thus, we can take up to 5 logarithms. Including the initial term, this means there are 6 well-defined terms.

28. D. Let $M = a_1a_2 + \cdots + a_9a_{10}$. The key idea is to realize that we can make the first seven terms 1 without changing the max value attainable. To see this, suppose in some arrangement a_1 is not equal to 1. Then, lowering a_1 by 1 and raising a_3 by 1 keeps the sum constant, but increases M by a_4. We then assume $a_1 = 1$ and continue inductively. Lowering a_k by 1 and raising a_{k+2} by 1 increases M by $a_{k+3} - a_{k-1}$. By the inductive hypothesis, $a_{k-1} = 1$, so this always at least increases M.

    In conclusion, we can assume $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a_7 = 1$. Then we're left with $6 + a_8 + a_8a_9 + a_9a_{10}$. If a_9 is fixed, this expression is maximized by setting $a_{10} = 1$. Thus, we're left with $6 + a_8 + a_8a_9 + a_9$, where $a_8 + a_9 = 32$. This is a quadratic in a_9: which is maximized for $a_9 = 16$. Thus, we get a final value of $6 + 16 + 256 + 16 = 294$.

29. A. If the first term is 4. Then the n^{th} term is $2^2 \times 2^{3(n-1)} = 2^{3n-1}$. The product of the first ten terms is then $2^{3(1+2+\cdots+10)-10} = 2^{155}$

30. A. Suppose $c > \frac{1}{4}$. Then

$$z(x) - x = (x^2 + c) - x = \left(x - \tfrac{1}{2}\right)^2 - \tfrac{1}{4} + c \geq c - \tfrac{1}{4} > 0.$$

    Similarly, we have

$$z\big(z(x)\big) - x = \Big(z\big(z(x)\big) - z(x)\Big) + (z(x) - x) \geq 2(c - \frac{1}{4})$$

    More generally, we have $z^n(x) - x \geq n\left(c - \frac{1}{4}\right)$. As n grows, this quantity grows to infinity since $c > \frac{1}{4}$. Thus, the sequence is unbounded.

    Now I claim that the sequence *is* bounded for $c = \frac{1}{4}$. This implies that $\frac{1}{4}$ is the answer to the problem. Suppose $c = \frac{1}{4}$; I claim that every term of the sequence is less than $\frac{1}{2}$. Clearly this is true for $z(0)$. Now suppose it's true for some term r in the sequence, so $0 < r < \frac{1}{2}$ (clearly every term of the sequence is positive). Then $z(r) = r^2 + \frac{1}{4} < \left(\frac{1}{2}\right)^2 + \frac{1}{4} = \frac{1}{2}$. By induction, we are done.