Geometry Hustle Answers

1. 525π
2. 441π
3. 8
4. 4
5. $125/2$ or 62.5 or $62\frac{1}{2}$
6. $17/2$ or 8.5 or $8\frac{1}{2}$
7. 35
8. 5
9. The open interval $(3, \sqrt{65})$
10. $65/2$ or $32\frac{1}{2}$ or 32.5
11. $1/3$
12. 30
13. $\sqrt{3} + 2\sqrt{2}$ or $2\sqrt{2} + \sqrt{3}$
14. 8
15. 4
16. 7
17. $1/4$ or 0.25
18. 7
19. 5
20. 27
21. 3
22. abcd
23. Proof by Construction or Constructive Proof
24. 4
25. 5

Geometry Hustle Solutions

(1) $\boldsymbol{V = \pi r^2 h}$. Plugging in the values gives a volume of $\boldsymbol{525\pi}$.

(2) As $n \to \infty$, the n-gon approaches a circle, thus the apothem length is the radius of the circle, so the area approaches 441π.

(3) It is clear that the maximum with 3 planes is 8 regions.

(4) $HB = 4$ by power of a point.

(5) Applying the formula $\frac{1}{2}ab\sin C$, we get that $[KIM] = \frac{1}{2}(10)(25)\sin 150° = \frac{1}{2}(10)(25)\frac{1}{2} = \frac{125}{2}$.

(6) By Shoelace Formula, the area is $17/2$.

(7) Aaron must take 4 steps up and 3 steps right, which gives $\binom{7}{3} = 35$ total moves.

(8) By triangle inequality on PAC, we see that $6 < PC < 14$. By triangle inequality on PCK, we see that $1 < PC < 13$. Thus, $6 < PC < 13$, which gives $\max(PC) = 12$ and $\min(PC) = 7$, $12 - 7 = 5$.

(9) By triangle inequality, $x > 3$. By Pythagora's Inequality, $x < \sqrt{65}$. Thus $(3, \sqrt{65})$.

(10) The area of the triangle is $\frac{1}{2}(5)(13)\sin K$ where angle K is across from side length k. The constraint is simply the triangle inequality. Thus the maximum area is when $\sin K = 1$ or when this is a right triangle. The area is $65/2$.

(11) Without loss of generality, let C lie on the positive x-axis, A lie on the positive y-axis, and M lie on the positive z-axis. The satisfactory tetrahedron has vertices $C = (1,0,0)$, $A = (0,2,0)$, and $M = (0,0,1)$. The volume is base $[COM] = 0.5$ times height $OA = 2$ over 3, which evaluates to $1/3$.

(12) An equilateral quadrilateral is a rhombus. A rhombus with diagonals 5 and 12 has area 30.

(13) $[SUN]$ is $2\sqrt{3}$ with probability $1/2$ and $4\sqrt{2}$ with probability $1/2$. Thus the expected value of $[SUN]$ is $\sqrt{3} + 2\sqrt{2}$.

(14) Note that NE can take on all lengths from 0 to 8. To visualize this, take the tangent from S to circle I. This is when N and E are the same point ($NE = 0$). Moving E along the circle to H' such that HIH' is a diameter ($NE = 8$), we see that the length of NE grows in a continuous manner between these endpoints. Note that $IN = IE = 4$, so INE is isosceles. Thus we maximize the value of $\frac{1}{2}(IN)(IE)\sin\angle NIE = 8\sin\angle NIE$. The maximum value for sine is 1, which occurs when $m\angle NIE = 90°$ or $NE = 4\sqrt{2}$, which is valid since $0 < 4\sqrt{2} < 8$.

(15) Let the leg lengths of a right triangle be a, b and hypotenuse be c. The romi ratio is $\frac{c^2}{ab/2} = \frac{2(a^2+b^2)}{ab}$. Note that the AM-GM inequality states that for non-negative numbers

x, y: $\frac{x+y}{2} \geq \sqrt{xy}$. This is equivalent, for $x, y \neq 0$, $\frac{x+y}{\sqrt{xy}} \geq 2$. Let $x = a^2$ and $y = b^2$, and apply AM-GM to get $\frac{a^2+b^2}{ab} \geq 2$ which multiplying both sides by 2, we get $\frac{2(a^2+b^2)}{ab} \geq 4$ as desired.

(16) Plugging in the values into Apollonius's Identity gives the following. $7^2 + 9^2 = 2x^2 + \frac{8^2}{2}$. Solving for x gives $KN = 7$.

(17) $[SRT]$ is half the area of STY since they share an altitude. $[ROT]$ is half the area of SRT by the same logic. Thus $\frac{[ROT]}{[STY]} = \frac{1}{4}$.

(18) KN is the hypotenuse of right triangles KON and KGN. Let $KG = x$. $x^2 + 36 = 4 + 81 \Rightarrow x = 7$.

(19) Let EV lie tangent to the semicircle at point A. Since tangents from the same point have equal length, $EA = EF = 4$. Additionally we can express $SV = VA = n$. Thus $VT = 4 - n$. Using Pythagorean Theorem on VET, we get $n^2 - 8n + 16 + 16 = n^2 + 8n + 16$. Solving, we get $n = 1$. We are solving for $EV = n + 4 = 5$.

(20) By similar triangles, $HM = 9$. Since HAM is a right triangle, $[HAM] = \frac{1}{2}(6)(9) = 27$.

(21) i. (Squaring a circle) and iii. (Trisecting an angle), are 2 canonical examples of impossible constructions, everything else is possible.

(22) This is a biconditional, so by definition $X \rightarrow Y$ and its converse must be true. Since the inverse is the contrapositive of the converse, and contrapositives hold the same truth value as the statement, the inverse of $X \rightarrow Y$ must be true. Thus, all abcd must be true.

(23) This is the definition of a proof by construction or constructive proof.

(24) All squares and rectangles are cyclic. An equilangular rhombus is the same thing as a square. Isosceles trapezoids are cyclic. Only the kite is not necessarily cyclic. Therefore, i, ii, iii, iv are cyclic.

(25) The tangent implies that SRY is a right triangle, so $STOR$ is inscribed in a semi-circle with radius 5. Thus, $STOR$ is an isosceles trapezoid, so TX is equal to the radius, which is 5.