2019 MU ALPHA THETA NATIONAL CONVENTION
MU AREA AND VOLUME
For all questions below, the answer E. NOTA means “None of these answers”.

1. B
Letting \( x \) denote the side length of each square cut from the corners. The volume of the resulting box is \( V(x) = x(10 - 2x)(16 - 2x) = 4x^3 - 52x^2 + 160x \). Then \( V'(x) = 12x^2 - 104x + 160 = 4(3x^2 - 26x + 40) = 4(3x - 20)(x - 2) = 0 \) when \( x = 20/3 \) or \( x = 2 \). The former is outside of the domain of \( V \) so \( x = 2 \) is the critical value. Noting \( V''(2) > 0 \), a minimum occurs when \( x = 2 \) and \( V(2) = 2(6)(12) = 144 \) cubic inches.

2. C
By symmetry, consider \( \int_{0}^{\infty} e^{2x} dx = \lim_{R \to \infty} \int_{R}^{0} e^{2x} dx = \lim_{R \to \infty} \frac{1}{2} e^{2x} \bigg|_{0}^{R} = \frac{1}{2} - 0 = \frac{1}{2} \)
Thus, the required area is \( 2(1/2) = 1 \).

3. B
Let \( D \) be the diameter of the larger sphere, \( s \) the side length of the cube, and \( d \) the diameter of the smaller sphere. Then \( \sqrt{3}s = D \) and \( s = d \), thus \( \sqrt{3}(d) = D \). Squaring both sides, \( 3(d^2) = D^2 \), so the ratio of the squares of the diameters is 3:1. This is also the ratio of the surface areas.

4. C
Let \((x, y)\) be a vertex of the rectangle on the semicircle. Then the area of the rectangle is \( A(x) = 2\sqrt{32} - x^2 \). Then \( A'(x) = 2\sqrt{32} - x^2 - \frac{2x^2}{\sqrt{32} - x^2} = 0 \) so \( x = 4 \). Testing \( x = 0 \) and \( x = 5 \) shows this is a maximum by the first derivative test. So \( A(4) = 32 \).

5. D
\( 3 = 2 + 2 \sin(\theta) \) yields \( \theta = \frac{\pi}{6}, \frac{5\pi}{6} \). The desired area is then
\[
\frac{1}{2} \int_{\pi/6}^{5\pi/6} ((2 + 2 \sin(\theta))^2 - 3^2)d\theta = \frac{1}{2} \int_{\pi/6}^{5\pi/6} (-5 + 8 \sin(\theta) + 4 \sin^2(\theta))d\theta
\]
\[
= \frac{1}{2} \int_{\pi/6}^{5\pi/6} (-5 + 8 \sin(\theta) + 2(1 - \cos(2\theta)))d\theta
\]
\[
= \frac{1}{2} \int_{\pi/6}^{5\pi/6} (-3 + 8 \sin(\theta) - 2 \cos(2\theta))d\theta
\]
\[
= \frac{1}{2} \left(-3 \left(\frac{5\pi}{6}\right) - 8 \left(-\frac{\sqrt{3}}{2}\right) + \frac{\sqrt{3}}{2} + 3 \left(\frac{\pi}{6}\right) + 8 \left(\frac{\sqrt{3}}{2}\right) + \frac{\sqrt{3}}{2}\right) = \frac{9\sqrt{3}}{2} - \pi
\]

6. B
Let the point on the curve be denoted by \((r, 3 - r^2)\). The equation of the tangent line is \( y = -2r(x - r) + 3 - r^2 = -2rx + 3 + r^2 \). The intercepts of this line are \((0, 3 + r^2)\) and \((3 + r^2, 0)\). The area of the triangle is \( A(r) = \frac{(3 + r^2)^2}{4r} \) and \( A'(r) = 0 \) implies \( 2(3 + r^2)(2r)(4r) = 4(3 + r^2)^2 \) for which \( r = 1 \). The minimum area is then \( A(1) = 4 \).

7. B
The volume using the disk method is \( \pi \int_{0}^{1} \left(\frac{3}{4}x + 6\right)^2 - 3^2 \) dx = 48\pi.
Also, the volume can be obtained by taking a large cone and subtracting away a similar cone and a cylinder: \( \frac{1}{3}\pi(6)^2 (8) - \pi(3)^2 (4) - \frac{1}{3}\pi(3)^2 (4) = 48\pi \).
8. B On this interval, the sine curve is greater. The desired area is 
\[ \int_{\pi/4}^{5\pi/4} (\sin(x) - \cos(x)) \, dx \]
\[ = -\cos(x) - \sin(x) \bigg|_{\pi/4}^{5\pi/4} = - \left( -\frac{1}{\sqrt{2}} \right) - \left( -\frac{1}{\sqrt{2}} \right) - \left( -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right) = \frac{4}{\sqrt{2}} = 2\sqrt{2}. \]

9. A Using the disk method, the volume is 
\[ \pi \int_{\pi/4}^{5\pi/4} ((\sin x + 1)^2 - (\cos x + 1)^2) \, dx \]

10. D The region formed is the interior of a square whose diagonal is length 12. The area is then \( \frac{1}{2}(12)(12) = 72 \) square units.

11. A Using shells, the required volume is 
\[ 2\pi \int_{0}^{\pi/4} (\sin x + 1 - (\cos x + 1)^2) \, dx \]

12. A The actual area is 
\[ \int_{\pi/2}^{2} \frac{2}{(x+1)(x+3)} \, dx = \int_{0}^{\pi/2} \frac{1}{x+1} - \frac{1}{x+3} \, dx \]
using partial fraction decomposition. This equals \( \ln(3) - \ln(1) - \ln(5) + \ln(3) = 2\ln(3) - \ln(5) = \ln \left( \frac{9}{5} \right) \).

13. B Let \( x \) and \( y \) be the two lengths that form the perimeter such that \( x + y = L \). Then the two side lengths are \( \frac{x}{4} \) and \( \frac{y}{4} \) and the sum of squares of areas is 
\[ A = \left( \frac{x}{4} \right)^2 + \left( \frac{y}{4} \right)^2 . \]
Substituting \( y = L - x \), we have 
\[ A(x) = \left( \frac{x}{4} \right)^2 + \left( \frac{L-x}{4} \right)^2 . \] The derivative is 
\[ A'(x) = \frac{x}{2} \left( \frac{1}{4} \right) + \frac{L-x}{2} \left( -\frac{1}{4} \right) = \frac{x}{4} - \frac{L}{8} = 0 \] when \( x = \frac{L}{2} \). Since \( A''(x) = \frac{1}{4} > 0 \) the value \( x = \frac{L}{2} \) will create a minimum. Thus, 
\[ A \left( \frac{L}{2} \right) = 2 \left( \frac{L}{8} \right)^2 = \frac{L^2}{32} . \]

14. B We find \( \Delta x = \frac{\pi}{4}, f(0) = 0, f \left( \frac{\pi}{4} \right) = \frac{1}{2}, f \left( \frac{\pi}{2} \right) = 1, f \left( \frac{3\pi}{4} \right) = \frac{1}{2}, f(\pi) = 0 . \)
The approximation of the area is 
\[ \frac{\pi}{2} \left( 0 + 2 \left( \frac{1}{2} \right) + 2(1) + 2 \left( \frac{1}{2} \right) + 0 \right) = \frac{\pi}{2} . \] Interestingly, this is the exact area of the region!
15. B A regular octahedron is composed of two square pyramids. By drawing in a diagonal of the base and a height of one pyramid, we see that half the diagonal is of length $3\sqrt{2}$ and thus the height is also $3\sqrt{2}$ using a 45-45-90 triangle. The volume of the octahedron is then $2 \cdot \frac{1}{3} \cdot (6)^2 \cdot 3\sqrt{2} = 72\sqrt{2}$.

16. C Each of the eight faces is an equilateral triangle of side length 6. The area of one of these faces is $\frac{\sqrt{3}}{4} (6)^2 = 9\sqrt{3}$. Multiplying by 8, the surface area is $72\sqrt{3}$.

17. D The area of $R$ is $\int_0^2 x^3 dx = \frac{1}{4} (2)^4 = 4$. Thus, $\int_0^a x^3 dx = 2$ and so $\frac{a^4}{4} = 2$ and Hence $a^4 = 8$ so $a = 8^{1/4} = 2^{3/4}$.

18. B The area bounded by a polar curve is $\frac{1}{2} \int_{\theta_1}^{\theta_2} r^2 d\theta$. Here, $r^2 = 9\cos(2\theta)$. Since one solution to $9 \cos(2\theta) = 0$ is $\theta = \frac{\pi}{4}$, we can integrate on $[0, \frac{\pi}{4}]$ and multiply the answer by 4: $4(1/2)\int_0^{\pi/4} 9\cos(2\theta) d\theta = 18 \int_0^{\pi/4} \cos(2\theta) d\theta$. Note the graph repeats after $\theta = \pi$.

19. B Let $A$ be the area of the largest square, $S$. Each shaded square’s area is $\frac{1}{4}$ the previous, so the total shaded area is $\frac{A}{4} \left(1 + \left(\frac{1}{4}\right) + \left(\frac{1}{4}\right)^2 + \cdots\right) = \frac{\frac{A}{4}}{1 - \frac{1}{4}} = \frac{A}{3}$. Thus, 1/3 of the area of the largest square is shaded.

20. D The sides of the square are $2y = 2\sqrt{r^2 - x^2}$ and the area of the square is $4(r^2 - x^2)$. Hence the volume is $\int_{-r}^r 4(r^2 - x^2) dx = 8(r^2 x - \frac{x^3}{3})\bigg|_{-r}^r = \frac{16r^3}{3}$.

21. D We consider all points exterior to the square of distance 1 from the nearest point on the square. This forms the set $T$ of points as shown. The enclosed area is then $(3)(3) + 4(1)(3) + 4 \cdot \frac{\pi}{4} (1)^2 = 21 + \pi$.

22. C The surface area of the original 3x3x3 cube is 6(3)(3) = 54. Since the new cube covers one of the faces of a 1x1x1 cube, the original area goes down to 53. Of the six faces of the 1x1x1 cube, one is obscured so of its faces, only 5 contribute. Thus, the new surface area is 53 + 5 = 58.
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23. E  By symmetry, consider \( \int_0^k |\cos(x)| \, dx = 3 \). We see that \( \int_0^{\frac{\pi}{2}} \cos(x) \, dx = 1 \). This means that \( \int_0^{\frac{3\pi}{2}} |\cos(x)| \, dx = 3 \) so \( k = \frac{3\pi}{2} \).

24. C  \( 2 \int_0^6 (f(x) + 2) \, dx = 2 \int_0^6 f(x) \, dx + 2 \int_0^6 2 \, dx = 2 \left( \frac{1}{2} (2)(6) - \frac{1}{2} \pi(2)^2 \right) + 2(12) \). This simplifies to \( 36 - 4\pi \).

25. B  Letting \( x \) be the vertex angle, the area is \( \frac{1}{2} (8)(8) \sin x = 32 \sin x \). The maximum is 32 when \( x = 90 \) degrees.

26. B  Writing \( A = wl \), we see \( \frac{dA}{dt} = \frac{dw}{dt} l + \frac{dl}{dt} w = (2)(8) - (4)(5) = -4 \text{cm}^2/\text{sec} \). This means the area is decreasing at 4 square cm per second.

27. A  Letting \( x \) be a side length of the base and \( h \) the height of the prism, we have \( x^2 h = V \) so \( h = \frac{V}{x^2} \). The surface area \( S = 4xh + 2x^2 \) so \( S(x) = \frac{4V}{x} + 2x^2 \). Then \( S'(x) = 4x - \frac{4V}{x^2} = 0 \) so \( x = \sqrt[3]{V} \). This is a minimum since \( S''(x) = 4 + \frac{8V}{x^3} > 0 \) for \( x = \sqrt[3]{V} \). The surface area is then \( S \left( \sqrt[3]{V} \right) = \frac{4V}{\sqrt[3]{V}} + 2V^{\frac{2}{3}} \)
or simply \( 6V^{\frac{2}{3}} \).

28. D  We are given \( \frac{dv}{dt} = -\pi \). Differentiating \( V = \frac{4}{3} \pi r^3 \), \( \frac{dv}{dt} = 4\pi r^2 \frac{dr}{dt} \) and when \( r = 1 \), we have \( \frac{dr}{dt} = -\frac{1}{4 \text{min}} \).

29. C  The surface area is \( 2\pi \int_4^4 yds \) where \( ds = \sqrt{1 + (y')^2} \, dx \). Here, \( ds = \sqrt{1 + \left( \frac{-x}{\sqrt{16-x^2}} \right)^2} \, dx = \frac{4}{\sqrt{16-x^2}} \, dx \). Thus, \( 2\pi \int_{-4}^4 \sqrt{16-x^2} \frac{4}{\sqrt{16-x^2}} \, dx = 8\pi \int_{-4}^4 dx = 8\pi(4 - (-4)) = 64\pi \). Alternatively, this is just the surface area of a sphere of radius 4, which has area \( 4\pi(4)^2 = 64\pi \).

30. C  One way to find this volume is the absolute value of the determinant of the matrix with the given vectors as columns: \[ \begin{vmatrix} 3 & 2 & 1 \\ -1 & 3 & 0 \\ 2 & 2 & 5 \end{vmatrix} \]. This determinant equals \( 3(15 - 0) - 2(-5 - 0) + 1(-2 - 6) = 45 + 10 - 8 = 47 \).