2019 MU ALPHA THETA NATIONAL CONVENTION

MU AREA AND VOLUME

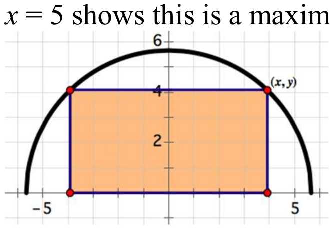
For all questions below, the answer E. NOTA means “None of these answers”.

1. B Letting x denote the side length of each square cut from the corners. The volume of the resulting box is $V(x) = x(10 - 2x)(16 - 2x) = 4x^3 - 52x^2 + 160x$. Then $V'(x) = 12x^2 - 104x + 160 = 4(3x^2 - 26x + 40) = 4(3x - 20)(x - 2) = 0$ when $x = 20/3$ or $x = 2$. The former is outside of the domain of V so $x = 2$ is the critical value. Noting $V''(2) > 0$, a minimum occurs when $x = 2$ and $V(2) = 2(6)(12) = 144$ cubic inches.

2. C By symmetry, consider $\int_{-\infty}^{0} e^{2x}dx = \lim_{R\to-\infty} \int_{R}^{0} e^{2x}dx = \lim_{R\to-\infty} \frac{1}{2}e^{2x}\Big|_{R}^{0} = \frac{1}{2} - 0 = \frac{1}{2}$. Thus, the required area is 2(1/2) = 1.

3. B Let D be the diameter of the larger sphere, s the side length of the cube, and d the diameter of the smaller sphere. Then $\sqrt{3}s = D$ and $s = d$, thus $\sqrt{3}(d) = D$. Squaring both sides, $3(d^2) = D^2$, so the ratio of the squares of the diameters is 3:1. This is also the ratio of the surface areas.

4. C Let (x, y) be a vertex of the rectangle on the semicircle. Then the area of the rectangle is $A(x) = 2x\sqrt{32 - x^2}$. Then $A'(x) = 2\sqrt{32 - x^2} - \frac{2x^2}{\sqrt{32-x^2}} = 0$ so $x = 4$. Testing $x = 0$ and $x = 5$ shows this is a maximum by the first derivative test. So $A(4) = 32$.



5. D $3 = 2 + 2\sin(\theta)$ yields $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$. The desired area is then

$$\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} ((2 + 2\sin(\theta))^2 - 3^2)d\theta = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (-5 + 8\sin(\theta) + 4\sin^2(\theta))d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (-5 + 8\sin(\theta) + 2(1 - \cos(2\theta)))d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (-3 + 8\sin(\theta) - 2\cos(2\theta)))d\theta$$

$$= \frac{1}{2}(-3\theta - 8\cos(\theta) - \sin(2\theta))\Big|_{\frac{\pi}{6}}^{\frac{5\pi}{6}}$$

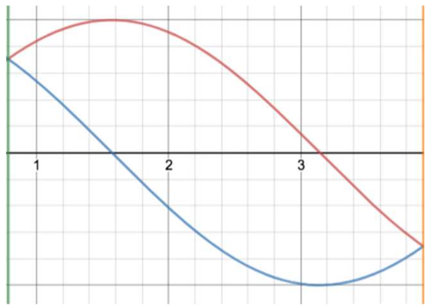
$$= \frac{1}{2}\left(-3\left(\frac{5\pi}{6}\right) - 8\left(-\frac{\sqrt{3}}{2}\right) + \frac{\sqrt{3}}{2} + 3\left(\frac{\pi}{6}\right) + 8\left(\frac{\sqrt{3}}{2}\right) + \frac{\sqrt{3}}{2}\right) = \frac{9\sqrt{3}}{2} - \pi.$$

6. B Let the point on the curve be denoted by $(r, 3 - r^2)$. The equation of the tangent line is $y = -2r(x - r) + 3 - r^2 = -2rx + 3 + r^2$. The intercepts of this line are $(0, 3 + r^2)$ and $(\frac{3+r^2}{2r}, 0)$. The area of the triangle is $A(r) = \frac{(3+r^2)^2}{4r}$ and $A'(r) = 0$ implies $2(3 + r^2)(2r)(4r) = 4(3 + r^2)^2$ for which $r = 1$. The minimum area is then $A(1) = 4$.

7. B The volume using the disk method is $\pi\int_0^4 \left(\left(-\frac{3}{4}x + 6\right)^2 - 3^2\right)dx = 48\pi$. Also, the volume can be obtained by taking a large cone and subtracting away a similar cone and a cylinder: $\frac{1}{3}\pi(6)^2(8) - \pi(3)^2(4) - \frac{1}{3}\pi(3)^2(4) = 48\pi$.

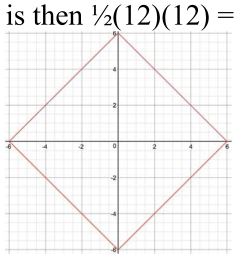
8. B On this interval, the sine curve is greater. The desired area is $\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin(x) - \cos(x))dx$

$= -\cos(x) - \sin(x)|_{\frac{\pi}{4}}^{\frac{5\pi}{4}} = -\left(-\frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) = \frac{4}{\sqrt{2}} = 2\sqrt{2}.$



9. A Using the disk method, the volume is $\pi \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} ((\sin x + 1)^2 - (\cos x + 1)^2)dx$

10. D The region formed is the interior of a square whose diagonal is length 12. The area is then ½(12)(12) = 72 square units.



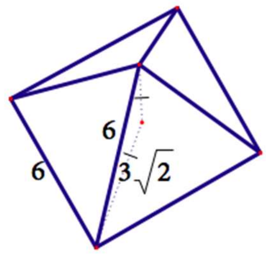
11. A Using shells, the required volume is $2\pi \int_{-1}^{0} (1 - x)e^{-x}dx.$ Using integration by parts, let $u = 1 - x, du = -dx$ and $dv = e^{-x}dx, v = -e^{-x}$. This yields,
$2\pi(-(1 - x)e^{-x})|_{-1}^{0} - 2\pi \int_{-1}^{0} e^{-x}dx = 2\pi(-1 + 2e) + 2\pi(1 - e) = 2\pi e.$

12. A The actual area is $\int_0^2 \frac{2}{(x+1)(x+3)} dx = \int_0^2 \frac{1}{x+1} - \frac{1}{x+3} dx$ using partial fraction decomposition. This equals $\ln(3) - \ln(1) - \ln(5) + \ln(3) = 2\ln(3) - \ln(5) = \ln\left(\frac{9}{5}\right).$

13. B Let x and y be the two lengths that form the perimeter such that $x + y = L.$ Then the two side lengths are $\frac{x}{4}$ and $\frac{y}{4}$ and the sum of squares of areas is $A = \left(\frac{x}{4}\right)^2 + \left(\frac{y}{4}\right)^2$.
Substituting $y = L - x$, we have $A(x) = \left(\frac{x}{4}\right)^2 + \left(\frac{L-x}{4}\right)^2$. The derivative is
$A'(x) = \frac{x}{2}\left(\frac{1}{4}\right) + \frac{L-x}{2}\left(-\frac{1}{4}\right) = \frac{x}{4} - \frac{L}{8} = 0$ when $x = \frac{L}{2}$. Since $A''(x) = \frac{1}{4} > 0$ the value $x = \frac{L}{2}$ will create a minimum. Thus, $A\left(\frac{L}{2}\right) = 2\left(\frac{L}{8}\right)^2 = \frac{L^2}{32}.$

14. B We find $\Delta x = \frac{\pi}{4}, f(0) = 0, f\left(\frac{\pi}{4}\right) = \frac{1}{2}, f\left(\frac{\pi}{2}\right) = 1, f\left(\frac{3\pi}{4}\right) = \frac{1}{2}, f(\pi) = 0.$
The approximation of the area is $\frac{\frac{\pi}{4}}{2}\left(0 + 2\left(\frac{1}{2}\right) + 2(1) + 2\left(\frac{1}{2}\right) + 0\right) = \frac{\pi}{2}.$
Interestingly, this is the exact area of the region!

15. B A regular octahedron is composed of two square pyramids. By drawing in a diagonal of the base and a height of one pyramid, we see that half the diagonal is of length $3\sqrt{2}$ and thus the height is also $3\sqrt{2}$ using a 45-45-90 triangle. The volume of the octahedron is then $2 \cdot \frac{1}{3} \cdot (6)^2 \cdot 3\sqrt{2} = 72\sqrt{2}$.



16. C Each of the eight faces is an equilateral triangle of side length 6. The area of one of these faces is $\frac{\sqrt{3}}{4}(6)^2 = 9\sqrt{3}$. Multiplying by 8, the surface area is $72\sqrt{3}$.

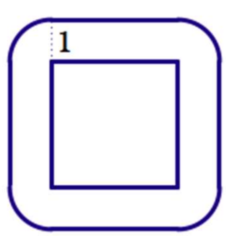
17. D The area of *R* is $\int_0^2 x^3 dx = \frac{1}{4}(2)^4 = 4$. Thus, $\int_0^a x^3 dx = 2$ and so $\frac{a^4}{4} = 2$ and Hence $a^4 = 8$ so $a = 8^{\frac{1}{4}} = (2^3)^{1/4} = 2^{\frac{3}{4}}$.

18. B The area bounded by a polar curve is ½ $\int_{\theta_1}^{\theta_2} r^2 d\theta$. Here, $r^2 = 9\cos(2\theta)$. Since one solution to $9\cos(2\theta) = 0$ is $\theta = \frac{\pi}{4}$, we can integrate on $[0, \frac{\pi}{4}]$ and multiply the answer by 4: $4(1/2)\int_0^{\frac{\pi}{4}} 9\cos(2\theta)\, d\theta = 18\int_0^{\frac{\pi}{4}} \cos(2\theta)\, d\theta$. Note the graph repeats after $\theta = \pi$.

19. B Let *A* be the area of the largest square, *S*. Each shaded square's area is ¼ the previous, so the total shaded area is $\frac{A}{4}\left(1 + \left(\frac{1}{4}\right) + \left(\frac{1}{4}\right)^2 + \cdots\right) = \frac{\frac{A}{4}}{1-\frac{1}{4}} = \frac{A}{3}$. Thus, 1/3 of the area of the largest square is shaded.

20. D The sides of the square are $2y = 2\sqrt{r^2 - x^2}$ and the area of the square is $4(r^2 - x^2)$. Hence the volume is $\int_{-r}^{r} 4(r^2 - x^2)dx = 8(r^2x - \frac{x^3}{3})\Big|_0^r = \frac{16r^3}{3}$.

21. D We consider all points exterior to the square of distance 1 from the nearest point on the square. This forms the set *T* of points as shown. The enclosed area is then $(3)(3) + 4(1)(3) + 4 \cdot \frac{\pi}{4}(1)^2 = 21 + \pi$.



22. C The surface area of the original 3x3x3 cube is 6(3)(3) = 54. Since the new cube covers one of the faces of a 1x1x1 cube, the original area goes down to 53. Of the six faces of the 1x1x1 cube, one is obscured so of its faces, only 5 contribute. Thus, the new surface area is 53 + 5 = 58.

23. E By symmetry, consider $\int_0^k |\cos(x)|dx = 3$. We see that $\int_0^{\frac{\pi}{2}} \cos(x)\, dx = 1$. This means that $\int_0^{\frac{3\pi}{2}} |\cos(x)|dx = 3$ so $k = \frac{3\pi}{2}$.

24. C $2\int_0^6 (f(x) + 2)dx = 2\int_0^6 f(x)dx + 2\int_0^6 2\, dx = 2\left(\frac{1}{2}(2)(6) - \frac{1}{2}\pi(2)^2\right) + 2(12)$. This simplifies to $36 - 4\pi$.

25. B Letting x be the vertex angle, the area is $\frac{1}{2}(8)(8)\sin x = 32\sin x$. The maximum is 32 when x = 90 degrees.

26. B Writing $A = wl$, we see $\frac{dA}{dt} = \frac{dw}{dt}l + \frac{dl}{dt}w = (2)(8) - (4)(5) = -4cm^2/sec$
This means the area is decreasing at 4 square cm per second.

27. A Letting x be a side length of the base and h the height of the prism, we have $x^2h = V$ so $h = \frac{V}{x^2}$. The surface area $S = 4xh + 2x^2$ so $S(x) = \frac{4V}{x} + 2x^2$. Then $S'(x) = 4x - \frac{4V}{x^2} = 0$ so $x = V^{\frac{1}{3}}$. This is a minimum since $S''(x) = 4 + \frac{8V}{x^3} > 0$ for $x = V^{\frac{1}{3}}$. The surface area is then $S\left(V^{\frac{1}{3}}\right) = \frac{4V}{V^{\frac{1}{3}}} + 2V^{\frac{2}{3}}$ or simply $6V^{\frac{2}{3}}$.

28. D We are given $\frac{dV}{dt} = -\pi$. Differentiating $V = \frac{4}{3}\pi r^3$, $\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$.
Thus, $-\pi = 4\pi r^2\frac{dr}{dt}$ and when r = 1, we have $\frac{dr}{dt} = -\frac{1}{4}\frac{ft}{min}$.

29. C The surface area is $2\pi\int_{-4}^4 y ds$ where $ds = \sqrt{1 + (y')^2}dx$. Here, $ds = \sqrt{1 + \left(\frac{-x}{\sqrt{16-x^2}}\right)^2}\, dx = \frac{4}{\sqrt{16-x^2}}dx$. Thus, $2\pi\int_{-4}^4 \sqrt{16 - x^2}\,\frac{4}{\sqrt{16-x^2}}dx = 8\pi\int_{-4}^4 dx = 8\pi(4 - -4) = 64\pi$. Alternatively, this is just the surface area of a sphere of radius 4, which has area $4\pi(4)^2 = 64\pi$.

30. C One way to find this volume is the absolute value of the determinant of the matrix with the given vectors as columns: $\begin{vmatrix} 3 & 2 & 1 \\ -1 & 3 & 0 \\ 2 & 2 & 5 \end{vmatrix}$. This determinant equals $3(15 - 0) - 2(-5 - 0) + 1(-2 - 6) = 45 + 10 - 8 = 47$.