1) Find $y(2)$ if $y(x)$ is a positive function that satisfies the differential equation $xy' = y + 1$ and $y(1) = 2018$.

(A) 2018 (B) 4037

(C) 2020 (D) 4039 (E) NOTA

Solution: $xy' = y + 1 \rightarrow \frac{y'}{y+1} = \frac{1}{x} \rightarrow \ln(y+1) = \ln(x) + C \rightarrow y + 1 = Cx \rightarrow 2019 = C \rightarrow y = 2019x - 1 \rightarrow y(2) = 4037$. B.

2) Which of the following relations satisfies the differential equation $(x + y)y' = x - y$ with $y = 0$ when $x = 1$?

(A) $x^2 - 2xy - y^2 = 1$ (B) $x^2 - x - y = 0$

(C) $x^2 - 2xy - y^2 - x^4 = 0$ (D) $x^2 - 2xy - y^2 - x = 0$ (E) NOTA

Solution: $(x + y)y' = x - y \rightarrow y' = \frac{x-y}{x+y} \rightarrow y' = \frac{1-\frac{y}{x}}{1+\frac{y}{x}}$. Let $v = \frac{y}{x} \rightarrow xv' + v = y' \rightarrow xv' + v = \frac{1-v}{1+v} \rightarrow xv' = \frac{1-v-v(1+v)}{1+v} \rightarrow \frac{1+v}{1-2v-v^2}v' = \frac{1}{x} \rightarrow -\frac{1}{2}\ln(1 - 2v - v^2) = \ln(x) + C \rightarrow \frac{1}{\sqrt{1-2v-v^2}} = Cx \rightarrow \frac{x}{\sqrt{x^2-2xy-y^2}} = Cx \rightarrow \frac{(1)}{\sqrt{1}} = C(1) \rightarrow C = 1 \rightarrow \frac{x}{\sqrt{x^2-2xy-y^2}} = x \rightarrow x^2 - 2xy - y^2 = 1$. A.

3) If $y' + 4x^3y = x^3$ and $y = 2$ when $x = 0$, what is y when $x = 1$?

(A) $\frac{4e+7}{4e}$ (B) $\frac{4e+5}{4e}$

(C) $\frac{e+7}{4e}$ (D) $\frac{e+5}{4e}$ (E) NOTA

Solution: $y' + 4x^3y = x^3 \rightarrow e^{x^4}y' + 4x^3e^{x^4}y = x^3e^{x^4} \rightarrow \left[e^{x^4}y\right]' = x^3e^{x^4} \rightarrow e^{x^4}y = \frac{1}{4}e^{x^4} + C \rightarrow y = \frac{1}{4} + Ce^{-x^4} \rightarrow 2 = \frac{1}{4} + C \rightarrow C = \frac{7}{4} \rightarrow y = \frac{1}{4} + \frac{7}{4}e^{-x^4} \rightarrow y(1) = \frac{1}{4} + \frac{7}{4e} = \frac{e+7}{4e}$. C.

4) If $y' + \frac{1}{x}y = y^3$, $y = 2$ when $x = 1$, and $y > 0$, what y when $x = \frac{1}{2}$?

(A) $\frac{16}{23}$ (B) $\frac{16}{9}$

(C) $\frac{4\sqrt{23}}{23}$ (D) $\frac{4}{3}$ (E) NOTA

Solution: $y' + \frac{1}{x}y = y^3 \rightarrow y^{-3}y' + \frac{1}{x}y^{-2} = 1$. Let $v = y^{-2} \rightarrow v' = -2y^{-3}y'$. So $y^{-3}y' + \frac{1}{x}y^{-2} = 1 \rightarrow -\frac{1}{2}v' + \frac{1}{x}v = 1 \rightarrow v' - \frac{2}{x}v = -2 \rightarrow \frac{1}{x^2}v' - \frac{2}{x^3}v = -\frac{2}{x^2} \rightarrow \left[\frac{1}{x^2}v\right]' = -\frac{2}{x^2} \rightarrow \frac{1}{x^2}v = \frac{2}{x} + C \rightarrow v = 2x + Cx^2 \rightarrow \frac{1}{y^2} = 2x + Cx^2 \rightarrow y^2 = \frac{1}{2x+Cx^2} \rightarrow 4 = \frac{1}{2+C} \rightarrow C = -\frac{7}{4} \rightarrow y^2 = \frac{4}{8x-7x^2} = \frac{4}{4-\frac{7}{4}} = \frac{16}{9} \rightarrow y = \frac{4}{3}$

D.

5) Find $y(-1)$ if $y(x)$ is the solution to the differential equation $y' = \frac{y}{x}$ and $y(1) = 2019$.

(A) -2019 (B) 2017

(C) 2019 (D) -2017 (E) NOTA

Solution: $y' = \frac{y}{x} \rightarrow \frac{1}{y}y' = \frac{1}{x} \rightarrow \ln(y) = \ln(x) + C \rightarrow \ln(2019) = \ln(1) + C = C \rightarrow \ln(y) = \ln(2019x) \rightarrow y = 2019x$. When $x = 2, y = 4038$ E.

6) Given that $M(x,y) + y\sec(x)\frac{dy}{dx} = 0$ is an exact differential equation, which of the following is a possible value of $M(x,y)$?

(A) $y\ln|\sec(x) + \tan(x)|$ (B) $y^2\sec^2(x)$

(C) $\frac{1}{2}y^2\sec(x)\tan(x) + \sec^2(x)$ (D) $\frac{1}{2}y^2\sec(x)$ (E) NOTA

Solution: The general form of an exact equation can be written as $F_x + F_y\frac{dy}{dx} = 0$ and has a solution $F(x,y) = C$. So $F_y = y\sec(x) \rightarrow F = \frac{1}{2}y^2\sec(x) + A(x) \rightarrow F_x = \frac{1}{2}y^2\sec(x)\tan(x) + A'(x)$. Only one of the above has such a form. C.

7) Using the correct answer to Question 6, solve the exact equation in Question 6 given the additional information that $y = 0$ when $x = 0$. Which of the following is a possible value of x when $y = 1$?

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{3}$

(C) $\frac{2\pi}{3}$ (D) $\frac{7\pi}{6}$ (E) NOTA

Solution: From above we see that $F(x,y) = \frac{1}{2}y^2\sec(x) + A(x)$ and that $A'(x) = \sec^2(x) \rightarrow A(x) = \tan(x)$. Therefore the general solution is $F(x,y) = \frac{1}{2}y^2\sec(x) + \tan(x) = C = 0$ based on $y = 0$ when $x = 0$. When $y = 1$ we have $\sec(x) + 2\tan(x) = 0 \rightarrow \sin(x) = -\frac{1}{2} \rightarrow x = \frac{7\pi}{6}$. D.

8) What is the general form of the solution to $y''' - y'' - 9y' + 9y = 0$? Assume y is a function of x.

(A) $y = C_1e^{-3x} + C_2e^x + C_3e^{3x}$ (B) $y = C_1e^{-x} + C_2e^x + C_3e^{3x}$

(C) $y = C_1e^{-3x} + C_2e^{-x} + C_3e^{3x}$ (D) $y = C_1e^{-9x} + C_2e^x$ (E) NOTA

Solution: The solutions will be of the form $y = e^{rx} \rightarrow y' = re^{rx} \rightarrow y'' = r^2e^{rx} \rightarrow y''' = r^3e^{rx}$. So $y''' - y'' - 9y' + 9y = 0 \rightarrow e^{rx}(r^3 - r^2 - 9r + 9) = 0 \rightarrow e^{rx}\left(r^2(r-1) - 9(r-1)\right) = 0 \rightarrow r = \pm 3, 1$. A.

9) What is the general form of the solution to $y'' + 4y = 0$? Assume y is a function of x.

(A) $y = C_1e^{-2x} + C_2e^{2x}$ (B) $y = C_1\cos(2x) + C_2\sin(2x)$

(C) $y = C_1e^{-4x} + C_2e^{4x}$ (D) $y = C_1\cos(4x) + C_2\sin(4x)$ (E) NOTA

Solution: Using the same technique as above, $r^2 + 4 = 0 \to r = \pm 2i$. Rather than using complex exponentials we use sine and cosine. B.

10) What is the general form of the solution to $y'' + 4y' + 4y = 0$? Assume y is a function of x.

(A) $y = C_1e^{-2x} + C_2e^{-2x}$ (B) $y = C_1e^{-2x} + C_2x^{-2}$

(C) $y = C_1e^{-2x} + C_2e^{2x}$ (D) $y = C_1e^{-2x} + C_2xe^{-2x}$ (E) NOTA

Solution: Using the same technique as above, $r^2 + 4r + 4 = (r+2)^2 = 0 \to r = -2$ with multiplicity two. Making the second solution linearly independent through multiplication by x does the trick. D.

11) Of the six functions listed below, which of the following sets of three will not result in a Wronskian of zero?

I. $y = \sin^2(2x)$	II. $y = \cos^2(2x)$	III. $y = \sin(4x)$
IV. $y = \sin(2x)$	V. $y = \cos(4x)$	VI. $y = 2$

(A) I, II, and VI (B) II, V, and VI

(C) III, IV, and V (D) I, V, and VI (E) NOTA

Solution: Because of the Pythagorean and Power Reducing identities, A, B, and D are all linearly dependent. So the answer is C.

12) A tank has pure water flowing into it at 12 liters per minute. The contents of the tank are kept thoroughly mixed, and the contents flow out at 10 liters per minute. Initially, the tank contains 10 kg of salt in 100 liters of water. If the tank can hold at most 1,000 liters of water, what will the amount of salt, in kg, in the tank be when the tank is full?

(A) $\frac{1}{100{,}000}$ (B) $\frac{1}{10{,}000}$

(C) $\frac{1}{1{,}000}$ (D) $\frac{1}{100}$ (E) NOTA

Solution: Let $S(t)$ be the amount of salt in the tank at time t in minutes At any given time the volume of the tank is $V(t) = 100 + 2t$ and therefore the concentration will be $\frac{S(t)}{100+2t}$. Finally the amount of salt that is leaving is equal to the concentration at that time multiplied by the rate it is leaving: $\frac{dS}{dt} = -10\frac{S(t)}{100+2t} \to \frac{1}{S}\frac{dS}{dt} = -\frac{10}{100+2t} \to \ln(S) = -5\ln(100+2t) + C$. We know $S(0) = 10$ so $\ln(10) =$

$-5\ln(100) + C \rightarrow \ln(10) = -10\ln(10) + C \rightarrow C = 11\ln(10)$. So $\ln(S) = -5\ln(100 + 2t) + 11\ln(10) \rightarrow S(t) = \frac{10^{11}}{(100+2t)^5}$. The tank will be full when $100 + 2t = 1000 \rightarrow S(t) = \frac{10^{11}}{(1000)^5} = \frac{1}{10000}$.
B.

13) A cat starts out at the origin and runs with a speed 2 along the positive y-axis in the positive direction. A dog starts out at the point $(9,0)$ and runs with a speed 4, always in the direction of the instantaneous location of the cat. The graph of which of the following equations coincides with the curve traced by the path of the dog?

(A) $y = -3\sqrt{x} + \frac{1}{9}x^{\frac{3}{2}} + 6$

(B) $y = 6\sqrt{x} - \frac{1}{9}x^{\frac{3}{2}} - 15$

(C) $y = 3\sqrt{x} - \frac{1}{9}x^{-\frac{3}{2}} - \frac{728}{81}$

(D) $y = 6\sqrt{x} - \frac{1}{9}x^{-\frac{3}{2}} - \frac{1457}{81}$

(E) NOTA

Solution: Denote the dog's path as $y(x)$, and note that $\frac{dy}{dx} = -\frac{2t-y}{x}$ since the slope of the path of the dog is always towards at the current location of the cat. There are too many variables here, so we need to get rid of time. To do that, note that $\frac{dy}{dx} = -\frac{2t-y}{x} \rightarrow xy' = y - 2t \rightarrow xy'' + y' = y' - 2\frac{dt}{dx} \rightarrow xy'' = -2\frac{dt}{dx}$. Next, note that the speed of the dog is the length of the arc per unit time (i.e. $\frac{ds}{dt} = 4$). The differential arc length is $ds = \sqrt{1 + (y')^2}dx$ and so $\frac{dt}{dx} = \frac{dt}{ds}\frac{ds}{dx} = \frac{1}{4}\sqrt{1 + (y')^2}$. Thus $xy'' = -\frac{1}{2}\sqrt{1 + (y')^2}$. Now we let $p(x) = y'$ and so $xp' = -\frac{1}{2}\sqrt{1 + p^2} \rightarrow \frac{p'}{\sqrt{1+p^2}} = -\frac{1}{2}\frac{1}{x} \rightarrow \sinh^{-1}(p) = -\frac{1}{2}\ln(x) + C \rightarrow p = y' = \sinh\left(-\frac{1}{2}ln(x) + C\right) = \frac{1}{2}\left(e^{-\frac{1}{2}ln(x)+C} - e^{\frac{1}{2}ln(x)-C}\right) = \frac{1}{2}\left(\frac{c}{\sqrt{x}} - \frac{\sqrt{x}}{c}\right)$. Now $y'(9) = 0$ since the dog starts out facing the cat at the origin. Therefore $\frac{1}{2}\left(\frac{c}{3} - \frac{3}{c}\right) = 0 \rightarrow c = -3$ (based on the graph). So $y' = \frac{1}{2}\left(-\frac{3}{\sqrt{x}} + \frac{\sqrt{x}}{3}\right) \rightarrow y = \frac{1}{2}\left(-6\sqrt{x} + \frac{2}{9}x^{\frac{3}{2}}\right) + D \rightarrow y = -3\sqrt{x} + \frac{1}{9}x^{\frac{3}{2}} + D$.
Finally we know the point $(9,0)$ is on the curve so $0 = -9 + 3 + D \rightarrow D = 6$. So $y = -3\sqrt{x} + \frac{1}{9}x^{\frac{3}{2}} + 6$.
A.

In problems 14-16 below, assume that the general solution can be written as $y = y_h + y_p$, where y_h is the general solution to the homogenous version of linear equation and y_p is referred to as the *particular solution*.

14) What is the particular solution of $y''' - y'' - 9y' + 9y = x^2$? Assume y is a function of x.

(A) $y_p = \frac{1}{9}x^2 - \frac{2}{9}x + \frac{7}{81}$

(B) $y_p = -\frac{1}{9}x^2 - \frac{2}{9}x + \frac{7}{81}$

(C) $y_p = -\frac{1}{9}x^2 + \frac{2}{9}x - \frac{16}{81}$

(D) $y_p = \frac{1}{9}x^2 - \frac{2}{9}x - \frac{7}{81}$

(E) NOTA

Solution: We use method of undetermined coefficients. The particular solution will be of the form $y_p = ax^2 + bx + c \rightarrow y_p' = 2ax + b \rightarrow y_p'' = 2a \rightarrow y_p''' = 0$. Therefore $y_p''' - y_p'' - 9y_p' - 9y_p = x^2 \rightarrow$

$-2a - 9(2ax + b) - 9(ax^2 + bx + c) = x^2 \to -9a = 1, -18a - 9b = 0, -2a - 9b - 9c = 0 \to a = -\frac{1}{9}, b = \frac{2}{9}, c = -\frac{16}{81}$. C.

15) What is the particular solution of $y'' + 4y = \cos(2x)$? Assume y is a function of x.

(A) $y_p = \frac{1}{4}x\sin(2x)$ (B) $y_p = \frac{1}{4}\sin(2x)$

(C) $y_p = \frac{1}{4}x\cos(2x)$ (D) $y_p = \frac{1}{4}\cos(2x)$ (E) NOTA

Solution: We use method of undetermined coefficients. Note that in this case the function generating the particular solution is $\cos(2x)$, which is linearly dependent with the homogenous solution. So our particular solution will be of the form $y_p = ax\cos(2x) + bx\sin(2x) \to y_p' = -2ax\sin(2x) + a\cos(2x) + 2bx\cos(2x) + b\sin(2x) \to y_p'' = -4ax\cos(2x) - 2a\sin(2x) - 2a\sin(2x) - 4bx\sin(2x) + 2b\cos(2x) + 2b\cos(2x)$. Therefore $y_p'' + 4y_p = \cos(2x) \to= -4ax\cos(2x) - 2a\sin(2x) - 2a\sin(2x) - 4bx\sin(2x) + 2b\cos(2x) + 2b\cos(2x) + 4ax\cos(2x) + 4bx\sin(2x) = \cos(2x) \to -2a\sin(2x) - 2a\sin(2x) + 2b\cos(2x) + 2b\cos(2x) = \cos(2x) \to a = 0, b = \frac{1}{4}$. A.

16) What is the particular solution of $y'' + 4y' + 4y = \sqrt[3]{x}e^{-2x}$? Assume y is a function of x.

(A) $-\frac{9}{28}x^{\frac{5}{3}}e^{-2x}$ (B) $-\frac{9}{28}x^{\frac{7}{3}}e^{-2x}$

(C) $\frac{9}{28}x^{\frac{5}{3}}e^{-2x}$ (D) $\frac{9}{28}x^{\frac{7}{3}}e^{-2x}$ (E) NOTA

Solution: For forcing functions of this form, we cannot use method of undetermined coefficients. We may use variation of parameters instead. First we set up the system $\begin{matrix} u_1'y_1 + u_2'y_2 = 0 \\ u_1'y_1' + u_2'y_2' = \sqrt[3]{x}e^{-2x} \end{matrix} \to \begin{matrix} u_1'e^{-2x} + u_2'xe^{-2x} = 0 \\ -2u_1'e^{-2x} + u_2'(1-2x)e^{-2x} = \sqrt[3]{x}e^{-2x} \end{matrix} \to \begin{matrix} 2u_1'e^{-2x} + 2u_2'xe^{-2x} = 0 \\ -2u_1'e^{-2x} + u_2'(1-2x)e^{-2x} = \sqrt[3]{x}e^{-2x} \end{matrix} \to u_2' = \sqrt[3]{x} \to u_1' = -x\sqrt[3]{x} = -x^{\frac{4}{3}} \to u_2 = \frac{3}{4}x^{\frac{4}{3}} \to u_1 = -\frac{3}{7}x^{\frac{7}{3}}$. Therefore $y_p = u_1y_1 + u_2y_2 = \left(-\frac{3}{7}x^{\frac{7}{3}}\right)e^{-2x} + \frac{3}{4}x^{\frac{4}{3}}xe^{-2x} = \frac{9}{28}x^{\frac{7}{3}}e^{-2x}$. D.

17) Let $y(x) = \sum_{n=0}^{\infty} a_nx^n$ be a solution to the differential equation $y'' + xy = e^x$ with $y(0) = y'(0) = 1$. Which of the following is a recursion relation satisfied by the sequence of coefficients $\{a_n\}$ for $n > 0$?

(A) $a_{n+2} = \frac{1}{n!} - \frac{a_n}{(n+2)(n+1)}$ (B) $a_{n+2} = \frac{1}{n!} - \frac{a_{n-1}}{(n+2)(n+1)}$

(C) $a_{n+2} = \frac{1}{(n+2)!} - \frac{a_n}{(n+2)(n+1)}$ (D) $a_{n+2} = \frac{1}{(n+2)!} - \frac{a_{n-1}}{(n+2)(n+1)}$ (E) NOTA

Solution: $y = \sum_{n=0}^{\infty} a_nx^n \to y' = \sum_{n=1}^{\infty} na_nx^{n-1} \to y'' = \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2}$. So $y'' + xy = e^{x^2} \to \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} + \sum_{n=0}^{\infty} a_nx^{n+1} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \to \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n +$

$\sum_{n=1}^{\infty} a_{n-1}x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} \rightarrow 0 = 2a_2 - 1 + \sum_{n=1}^{\infty}\left((n+2)(n+1)a_{n+2} + a_{n-1} - \frac{1}{n!}\right)x^n$. So we know since each term must be zero that $a_2 = \frac{1}{2}$ and $(n+2)(n+1)a_{n+2} + a_{n-1} - \frac{1}{n!} = 0 \rightarrow a_{n+2} = \frac{1}{(n+2)!} - \frac{a_{n-1}}{(n+2)(n+1)}$. D.

18) Consider the series solution to $y'' + xy = e^x$ with $y(0) = y'(0) = 1$ as described in Question 17 above. Find $a_0 + a_1 + a_2 + a_3 + a_4 + a_5$.

(A) $\frac{173}{120}$ (B) $\frac{35}{24}$

(C) $\frac{293}{120}$ (D) $\frac{59}{24}$ (E) NOTA

Solution: We know from the initial condition that $a_0 = 1$ and $a_1 = 1$. We also saw that $a_2 = \frac{1}{2}$. We need to use the recursion to get the last two terms: $a_{n+2} = \frac{1}{(n+2)!} - \frac{a_{n-1}}{(n+2)(n+1)} \rightarrow a_3 = \frac{1}{6} - \frac{1}{6} = 0, a_4 = \frac{1}{24} - \frac{1}{12} = -\frac{1}{24}, a_5 = \frac{1}{120} - \frac{\frac{1}{2}}{20} = -\frac{1}{60}$. Therefore $1 + 1 + \frac{1}{2} + 0 - \frac{1}{24} - \frac{1}{60} = \frac{293}{120}$. C.

19) Let $y(x) = \sum_{n=0}^{\infty} a_n x^n$ be a solution to the differential equation $(x^2 - 2x + 2)y'' + xy = 0$ with $y(0) = y'(0) = 1$. Find the radius of convergence of this series.

(A) 1 (B) $\sqrt{2}$

(C) 2 (D) ∞ (E) NOTA

Solution: Written a different way, $(x^2 - 2x + 2)y'' + xy = 0 \rightarrow y'' + \frac{x}{x^2-2x+2}y = 0$. This equation will have a singular point wherever $x^2 - 2x + 2 = 0 \rightarrow x = 1 \pm i$. Since the series is expanded about zero, the radius of convergence is precisely the distance from the origin to the singular point. $|1 \pm i| = \sqrt{2}$. B.

20) What are the eigenvalues of $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & 1 \\ -4 & 0 & 1 \end{bmatrix}$?

(A) $\{1, 0, -3\}$ (B) $\{-1, 0, 3\}$

(C) $\{1, 0, 3\}$ (D) $\{-1, 0, -3\}$ (E) NOTA

Solution: $\begin{vmatrix} 1-\gamma & 0 & -1 \\ 2 & -\gamma & 1 \\ -4 & 0 & 1-\gamma \end{vmatrix} = -\gamma(1-\gamma)^2 + 4\gamma = 0 \rightarrow \gamma(4 - (1-\gamma)^2) = 0 \rightarrow \gamma = -1, 0, 3$. B.

21) What are the eigenvectors of $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & 1 \\ -4 & 0 & 1 \end{bmatrix}$?

(A) $\begin{bmatrix}1\\4\\2\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}1\\0\\-2\end{bmatrix}$ (B) $\begin{bmatrix}1\\4\\2\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}1\\4\\-2\end{bmatrix}$

(C) $\begin{bmatrix}1\\-4\\2\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}1\\0\\-2\end{bmatrix}$ (D) $\begin{bmatrix}1\\-4\\2\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}1\\4\\-2\end{bmatrix}$ (E) NOTA

Solution: For each of the values determined in Question 20: (a) $\begin{bmatrix}1+1 & 0 & -1\\2 & 0+1 & 1\\-4 & 0 & 1+1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\rightarrow\begin{bmatrix}2&0&-1\\2&1&1\\-4&0&2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\rightarrow\begin{bmatrix}2&0&-1\\2&1&1\\-4&0&2\end{bmatrix}\begin{bmatrix}\mathbf{1}\\\mathbf{-4}\\\mathbf{2}\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$ (b) $\begin{bmatrix}1&0&-1\\2&0&1\\-4&0&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\rightarrow\begin{bmatrix}1&0&-1\\2&0&1\\-4&0&1\end{bmatrix}\begin{bmatrix}\mathbf{0}\\\mathbf{1}\\\mathbf{0}\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$ (c) $\begin{bmatrix}1-3&0&-1\\2&0-3&1\\-4&0&1-3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\rightarrow\begin{bmatrix}-2&0&-1\\2&-3&1\\-4&0&-2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\rightarrow\begin{bmatrix}2&0&-1\\2&1&1\\-4&0&2\end{bmatrix}\begin{bmatrix}\mathbf{1}\\\mathbf{0}\\\mathbf{-2}\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$ C.

22) If $x(t), y(t)$, and $z(t)$ are continuous differentiable functions satisfying the equations $\begin{bmatrix}x'=x-z\\y'=2x+z\\z'=-4x+z\end{bmatrix}$ then what is the most general solution vector $\begin{bmatrix}x(t)\\y(t)\\z(t)\end{bmatrix}$?

(A) $\begin{bmatrix}c_1e^t+c_3e^{-3t}\\c_2\\2c_1e^t-2c_3e^{-3t}\end{bmatrix}$ (B) $\begin{bmatrix}c_1e^{-t}+c_3e^{3t}\\c_2\\2c_1e^{-t}-2c_3e^{3t}\end{bmatrix}$

(C) $\begin{bmatrix}c_1e^t+c_3e^{-3t}\\-4c_1e^t+c_2\\2c_1e^t-2c_3e^{-3t}\end{bmatrix}$ (D) $\begin{bmatrix}c_1e^{-t}+c_3e^{3t}\\-4c_1e^{-t}+c_2\\2c_1e^{-t}-2c_3e^{3t}\end{bmatrix}$ (E) NOTA

Solution: By the properties of systems of differential equations, the answer is $\begin{bmatrix}x(t)\\y(t)\\z(t)\end{bmatrix}=c_1\begin{bmatrix}1\\-4\\2\end{bmatrix}e^{-t}+c_2\begin{bmatrix}0\\1\\0\end{bmatrix}+c_3\begin{bmatrix}1\\0\\-2\end{bmatrix}e^{3t}=\begin{bmatrix}c_1e^{-t}+c_3e^{3t}\\-4c_1e^{-t}+c_2\\2c_1e^{-t}-2c_3e^{3t}\end{bmatrix}$. D.

23) C: $y'''=\frac{d}{dx}(x+y-y^2)=1+y'-2yy'$. $y^{(4)}=y''-2yy''-2(y')^2$. Using the given information, we find that: $y''(0) = -2$, $y'''(0) = 4$, $y^{(4)}(0) = -8$.

Hence, y= $-1+x-x^2+\frac{2}{3}x^3-\frac{1}{3}x^4$. Thus, the sum is -2/3

24) What is the general form of the solution to $y' = 2x + \frac{1}{x}$?

(A) $y = x^2 - \frac{1}{x^2}$ (B) $y = x^2 + \ln(x)$

(C) $y = 2 - \frac{1}{x^2}$ (D) $y = 2 + \ln(x)$ (E) NOTA

Solution: None of these solutions have the free constant of integration needed to make it a general solution. E.

The Laplace Transform is extremely important in the study of differential equations. The Laplace transform of a function $f(t)$ is defined to be $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$.

25) Let $f(t)$ be a continuous, twice-differentiable function that does not grow faster than every exponential function. Find $\mathcal{L}\{e^{at}f(t)\}$ in terms of $\mathcal{L}\{f(t)\} = F(s)$.

(A) $F(s-a)$ (B) $F(a-s)$

(C) $\frac{F(s-a)}{s-a}$ (D) $\frac{F(a-s)}{a-s}$ (E) NOTA

Solution: $\int_0^\infty e^{-st}e^{at}f(t)dt = \int_0^\infty e^{-(s-a)t}f(t)dt = F(s-a)$. A.

26) Let $u(t-a) = \begin{cases} 0 & t < a \\ 1 & t > a \end{cases}$. Let $f(t)$ be a continuous, twice-differentiable function that does not grow faster than every exponential function. Find $\mathcal{L}\{u(t-a)f(t-a)\}$ in terms of $\mathcal{L}\{f(t)\} = F(s)$.

(A) $e^{-as}F(s)$ (B) $e^{as}F(s)$

(C) $e^{-as}F(s+a)$ (D) $e^{as}F(s+a)$ (E) NOTA

Solution: $\int_0^\infty e^{-st}u(t-a)f(t-a)dt = \int_a^\infty e^{-st}f(t-a)dt = \int_0^\infty e^{-s(t+a)}f(t)dt =$
$\int_0^\infty e^{-s(t+a)}f(t)dt = e^{-as}\int_0^\infty e^{-s}\ f(t)dt = e^{-as}F(s)$. A.

27) Let $f(t)$ be a continuous, twice-differentiable function that does not grow faster than every exponential function. Further, let $f(0) = f_0$ and $f'(0) = f_1$. Find $\mathcal{L}\{f''(t)\}$ in terms of these constants and $\mathcal{L}\{f(t)\} = F(s)$.

(A) $s^2F(s) - sf_0 + f_1$ (B) $s^2F(s) + sf_0 + f_1$

(C) $s^2F(s) - sf_0 - f_1$ (D) $-s^2F(s) + sf_0 + f_1$ (E) NOTA

Solution: $\int_0^\infty e^{-st} f''(t)dt =$ (via integration by parts with $u = e^{-s}$ and $dv = f''(t)dt) =$ $[e^{-s}\, f'(t)]_0^\infty + s\int_0^\infty e^{-st} f'(t)dt = -f_1 + s\int_0^\infty e^{-st} f'(t)dt =$ (via integration by parts with $u = e^{-st}$ and $dv = f'(t)dt) = -f_1 + s\left([e^{-st} f(t)]_0^\infty + s\int_0^\infty e^{-s}\, f(t)dt\right) = s^2\mathcal{L}\{f(t)\} - sf_0 - f_1$. C.

28) Evaluate: $\mathcal{L}\{t^n\}$ for any non-negative integer n and any $s > 0$.

(A) $\frac{(n-1)!}{s^n}$ (B) $\frac{(n-1)!}{s^{n+1}}$

(C) $\frac{n!}{s^n}$ (D) $\frac{n!}{s^{n+1}}$ (E) NOTA

Solution: $\int_0^\infty t^n e^{-s}\, f(t)dt = \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}dt$ using integration by parts. Therefore if $F_n(s) \equiv \mathcal{L}\{t^n\}$ then we have the recursion $F_n(s) = \frac{n}{s}F_{n-1}(s)$. Since we already know $F_0(s) = \mathcal{L}\{1\} = \frac{1}{s}$, we get immediately that $F_n(s) = \frac{n!}{s^{n+1}}$. D.

29) Let $f(t)$ be a continuous, twice-differentiable function that does not grow faster than every exponential function. Find $\mathcal{L}\left\{\int_0^t f(x)dx\right\}$ in terms of $\mathcal{L}\{f(t)\} = F(s)$.

(A) $F\left(F(s)\right)$ (B) $\frac{F(s)}{s}$

(C) $sF(s)$ (D) $\frac{F(s)}{s^2}$ (E) NOTA

Solution: Let $g(t) = \int_0^t f(x)dx$. Then $g'(t) = f(t) \rightarrow \mathcal{L}\{g'(t)\} = F(s) \rightarrow sG(s) = F(s) \rightarrow G(s) = \frac{F(s)}{s}$. B.

30) B: From the exponents, we know that the solutions to the auxillary equation are -2 and 1. Working backwards we have: (r+2)(r-1)=0 → $r^2 + r - 2 = 0$ →DE must be: y'' + y' – 2y = 0