ANSWERS

1. D
2. B
3. B
4. A
5. B
6. E $(\frac{26}{53}+\frac{15}{53}i)$
7. A
8. A
9. B
10. C
11. C
12. C
13. B
14. C
15. D
16. E (17)
17. D
18. B
19. B
20. B
21. C
22. D
23. C
24. E (225)
25. D
26. D
27. D
28. A
29. D
30. C

1. Faustine's share is $\$1000 \times \frac{3}{8} = \$375$.

2. We pick the last letter; there are 2 ways to do this. Then the other five letters can be arranged in 5! ways. Therefore the total number of ways is $2 \cdot 5! = 2 \cdot 120 = 240$.

3. Suppose w and h are the width and height of the rectangle. Then the measurements are $2w + h$ and $w + 2h$. Thus, $(2w + h) + (w + 2h) = 3w + 3h = 80 + 88 = 168$. Therefore $w + h = \frac{168}{3} = 56$, and so $2(w + h) = 2 \cdot 56 = 112$ is the perimeter.

4. The difference doesn't change when 4 is added to each number. Hence, the two "new" numbers have a difference of 20 and a ratio of 3. These numbers must be 10 and 30; therefore the original numbers were 6 and 26.

5. Let C, F, and K be the ages of Cantoraptor, Fermatdactyl, and Kovalevskasaurus in millions of years. Then $C + F + K = 360$ and $C = 2K$. So K million years ago, Cantoraptor was K million years old and Fermatdactyl was $0.5K$ million years old. Hence, $F = 1.5K$. Thus,

$$C + F + K = 2K + 1.5K + K = 4.5K = 360$$

which implies $K = \frac{360}{4.5} = 80$ and so $F - K = 1.5K - K = 0.5K = 0.5 \cdot 80 = 40$.

6. We multiply by the conjugate.

$$\frac{i^2 + 4i}{7i + 2} = \frac{4i - 1}{7i + 2} \cdot \frac{7i - 2}{7i - 2} = \frac{28i^2 - 15i + 2}{-49 - 4} = \frac{-26 - 15i}{-53} = \frac{26}{53} + \frac{15}{53}i.$$

7. The area of the triangle formed by two vectors is half of the magnitude of the cross product of the two vectors. Thus

$$\begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 4 & -1 & 1 \\ -2 & 1 & 3 \end{vmatrix} = (-1 \cdot 3 - 1 \cdot 1)\vec{\imath} - (4 \cdot 3 + 2 \cdot 1)\vec{\jmath} + (4 \cdot 1 - 2 \cdot 1)\vec{k} = -4\vec{\imath} - 14\vec{\jmath} + 2\vec{k}$$

is the cross product, and hence the area is

$$\frac{1}{2}\sqrt{4^2 + 14^2 + 2^2} = \frac{1}{2}\sqrt{2^2(2^2 + 7^2 + 1)} = \sqrt{54} = 3\sqrt{6}.$$

8. There are 20 plants which have both green peas and smooth peas, and there are 6 which have all three characteristics. Thus, there must be $20 - 6 = 14$ plants which have green peas and smooth peas and are not tall.

9. We want the sum of the coordinates to be 5; in other words, we want $x + y + z = 5$. So we solve the new system of equations

$$\begin{cases} x - y + z = 0 \\ x + y - z = 4 \\ x + y + z = 5. \end{cases}$$

Adding the first equations yields $2x = 4$ so that $x = 2$. Adding the last two equations yields $2x + 2y = 9$ and using the fact that $x = 2$, we get $y = 2.5$. Using these values in the first equation gives us $z = 0.5$. Now going back to the original system of equations, we substitute these values into the last equation, giving $x + 2y + 3z = 2 + 2 \cdot 2.5 + 3 \cdot 0.5 = 8.5 = k$.

10. We use the formula for the sum of the squares of the first n positive integers. We have

$$\lim_{n\to\infty}\frac{1^2+2^2+3^2+\cdots+n^2}{n^3}=\lim_{n\to\infty}\frac{n(n+1)(2n+1)/6}{n^3}=\lim_{n\to\infty}\frac{2n^3+3n^2+n}{6n^3}=\frac{2}{6}=\frac{1}{3}.$$

11. At the time of collision, the particles' x-coordinates must be equal for the same value of t, and their y-coordinates must be equal for this same value. Hence, setting $x_1(t)=x_2(t)$ gives us the equation $t^3-4t^2+t+6=0$, which factors into $(t+1)(t-2)(t-3)=0$. So the particles' x-coordinates are the same when $t=-1,2,3$. Now setting $y_1(t)=y_2(t)$ gives us $t^3-t^2-9t+9=0$, or $(t+3)(t-1)(t-3)=0$. So the particles' y-coordinates are equal for $t=-3,1,3$. Thus, both coordinates are equivalent only when $t=3$. At this time, we have $x_1(3)=18$ and $y_1(3)=28$. Hence, the particles collide at $(18,28)$.

12. We investigate two cases, N even and N odd. If N is even, then $48=N+\frac{N}{2}=\frac{3N}{2}$, and so $N=\frac{2}{3}\cdot 48=32$. If N is odd, then its second-largest factor k must satisfy $k\le\frac{N}{3}$. Then $48=N+k\le\frac{4N}{3}$, so that $N\ge\frac{3}{4}\cdot 48=36$. Thus, we only need to check the odd numbers from 37 to 47. We find that $37+1=38$, $39+13=52$, $41+1=42$, $43+1=44$, $45+15=60$, and $47+1=48$. Hence, the possible values of N are 32 and 47.

13. We use the washer method. The outer radius is the distance from the axis of revolution to the curve $y=e^x$; this is e^x+1. The inner radius is the distance from the axis of revolution to the curve $y=0$; this is $0+1=1$. Hence, the integral becomes

$$\pi\int_0^{\ln(2)}[(e^x+1)^2-1^2]\,dx=\pi\int_0^{\ln(2)}[e^{2x}+2e^x]\,dx=\pi\left(\frac{1}{2}e^{2x}+2e^x\right)\Big|_0^{\ln(2)}$$

so that the volume is

$$\pi\left(\frac{1}{2}e^{2\ln(2)}+2e^{\ln(2)}-\frac{1}{2}e^0-2e^0\right)=\pi\left(2+4-\frac{1}{2}-2\right)=\frac{7\pi}{2}.$$

14. The units digit of any integer raised to a nonnegative integer power is periodic with period 1, 2, or 4. Thus, we evaluate $f(k)$ for $k=0,1,2,3,4$. Since we are only considering the units digit, we may only consider the units digit of each term in the sum $f(k)$. Let $U_f(k)$ be the sum of the units digit of each term of $f(k)$. Thus,

$$U_f(0)=1\cdot 9=9$$
$$U_f(1)=1+2+\cdots+9=45$$
$$U_f(2)=1+4+9+6+5+6+9+4+1=45$$
$$U_f(3)=1+8+7+4+5+6+3+2+9=45$$
$$U_f(4)=1+6+1+6+5+6+1+6+1=33$$

Therefore, the possible units digits of $f(k)$ are 3, 5, and 9.

15. We compute the machine's expected production time. Call the expected value E. Then

$$60\%\cdot 9+30\%(5+E)+10\%(3+E)=E.$$

This equation simplifies to $7.2+0.4E=E$ so that $E=\frac{7.2}{0.6}=12$ minutes.

16. The prime factorization of 131327 is $7 \cdot 73 \cdot 257$. This suggests that $2^x + 1 = 2^8 + 1 = 257$ so that $2^y - 1 = 2^9 - 1 = 511 = 7 \cdot 73$. This is the only possibility since all factors of 131327 are 1, 7, 73, 257, 511, 1799, 18761, and 131327, and the only ones which are one more or one less than a power of 2 are 257 and 511. Thus $x + y = 8 + 9 = 17$.

17. First, note that if $k = 0$, the equation reduces to $2^{-x} = 3$ which has one solution, namely $x = \log_2 \frac{1}{3}$. Now assume that $k \neq 0$. Multiply both sides of the equation by 2^x and rearrange to get $k(2^x)^2 - 3(2^x) + 1 = 0$. This is quadratic in 2^x; the quadratic formula gives us

$$2^x = \frac{3 \pm \sqrt{9 - 4k}}{2k}.$$

If there is a single root, the discriminant will be zero. Hence, $9 - 4k = 0$ gives $k = \frac{9}{4}$. If $9 - 4k < 0$ then there are no roots. If $9 - 4k > 0$ then we must consider the fact that 2^x is never negative. So there will be a single root when the numerator $3 \pm \sqrt{9 - 4k}$ has one positive value. This happens when $3 < \sqrt{9 - 4k}$, or $9 < 9 - 4k$, which implies $k < 0$. Hence, the values of k which give a single root are $k \leq 0$ and $k = \frac{9}{4}$.

18. The volume of the box is 10^3 and the volume of the sphere is $\frac{4}{3}\pi \cdot 5^3$. Then the percentage taken by the air is the difference in the volumes divided by the volume of the box. This is

$$\frac{10^3 - \frac{4}{3}\pi \cdot 5^3}{10^3} = \frac{4 \cdot 5^3 \left(2 - \frac{\pi}{3}\right)}{10^3} = \frac{2 - \frac{\pi}{3}}{2} = 1 - \frac{\pi}{6}.$$

Since $\frac{25}{8} < \pi < \frac{22}{7}$, then $\frac{25}{48} < \frac{\pi}{6} < \frac{11}{21}$. This implies $\frac{10}{21} < 1 - \frac{\pi}{6} < \frac{23}{48}$. Using fractions which can be easily converted to percentages, we find that $\frac{19}{40} < \frac{10}{21} < 1 - \frac{\pi}{6} < \frac{23}{48} < \frac{24}{25}$. Hence, $47.5\% < 1 - \frac{\pi}{6} < 48\%$. The percentage to the nearest whole percent is 48%.

19. Let us consider the base of the triangle $\overline{AB}$. Since $y = x^2 - 3x - 1$ is continuous and differentiable, we may apply the Mean Value Theorem on the interval $[-2, 4]$:

$$\frac{y(4) - y(-2)}{4 + 2} = \frac{3 - 9}{6} = -\frac{6}{6} = -1.$$

This is the value which maximizes the triangle, since it is at this point that a tangent to the parabola is parallel to $\overline{AB}$, thereby creating the largest possible distance from $\overline{AB}$ to the parabola, and hence, the largest possible height of triangle ABC. Since $y(1) = -3$, we have $m + n = 1 - 3 = -2$.

20. The presence of π in the answer choices and the complexity of the equation given cause one to think of using polar coordinates. Indeed, since $r^2 = x^2 + y^2$ and $x = r\cos\theta$, we convert the equation into $(r^2 - r\cos\theta)^2 = r^2$. This simplifies to $r = 1 + \cos\theta$. Now we set up and evaluate an integral to find the polar area. The integral is

$$\frac{1}{2}\int_0^{2\pi} (1 + \cos\theta)^2 \, d\theta = \frac{1}{2}\int_0^{2\pi} (1 + 2\cos\theta + \cos^2\theta) \, d\theta = \frac{1}{2}\int_0^{2\pi} \left(\frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta\right) d\theta.$$

Thus, the antiderivative, and therefore the area, is

$$\frac{1}{2}\left(\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\right)\Big|_0^{2\pi} = \frac{1}{2}(3\pi) = \frac{3\pi}{2}.$$

21. Let the mean be $n-1$, the median be n, and the mode be $n+1$. Then the numbers must be $a, b, n, n+1, n+1$ where a and b are integers. The sum of these five numbers must be equal to five times the mean; hence, $a+b+3n+2=5n-5$ which becomes $a+b=2n-7$. Now, the largest b can be is $n-1$ so the smallest a can be is $2n-7-(n-1)=n-6$. Hence, the largest value of the range is $n+1-(n-6)=7$.

22.

$$v(t)=\frac{1}{2}\cos t-2\sin 2t$$

$$a(t)=\frac{-1}{2}\sin t-4\cos 2t \rightarrow a\left(\frac{\pi}{2}\right)=\frac{-1}{2}+4=\frac{7}{2}$$

23. Since the highest powered terms are the same it is the ratio of the coefficients of those terms, so 1/4

24. Since $2x$ and 28 are even, so is $y+z$. Hence either y and z are both even or both odd.
Case I, both even. If y and z are both even, there exists integers m and n such that $y=2m$ and $z=2n$. Hence, $2x+y+z=2x+2m+2n=28$ so that $x+m+n=14$. By the method of stars-and-bars, there are $\binom{14+3-1}{3-1}=\binom{16}{2}=\frac{16\cdot 15}{2}=120$ solutions in this case.
Case II, both odd. If y and z are both odd, there exists integers m and n such that $y=2m+1$ and $z=2n+1$. Hence, $2x+y+z=2x+2m+1+2n+1=28$ so that $2x+2m+2n=26$, which implies $x+m+n=13$. There are $\binom{13+3-1}{3-1}=\binom{15}{2}=\frac{15\cdot 14}{2}=105$ solutions.
Therefore, there is a total of $120+105=225$ solutions.

25. Let the three numbers omitted be $n-1$, n, and $n+1$. The sum of the first 2019 positive integers is $\frac{2019\cdot 2020}{2}=2019\cdot 1010$. The value that Najifa calculated was

$$\frac{2019\cdot 1010-(n-1)-n-(n+1)}{2019}=\frac{2019\cdot 1010-3n}{2019}.$$

For this to be an integer, the numerator must be divisible by 2019; this happens at two different values, when $n=673$ and 1346. In the first case, the sum of the digits is 48. In the second, 42. The maximum is 48.

26. Clearly, 2, 3, 5, and 7 are splendiferous. Suppose n is a two-digit prime. Then n has two distinct digits, for otherwise it would be divisible by 11 and not prime. Moreover, both digits must themselves be prime. Since 2 and 5 can only appear as leading digits, the only two-digit splendiferous numbers are 23, 37, 53, and 73. The first and last two digits of a three-digit splendiferous number are also splendiferous; hence, they are one of 23, 37, 53, or 73. Carefully pairing these, we find that 237, 373, 537, and 737 are candidates for three-digit splendiferous numbers. Of these, only 373 is prime (237 and 537 are divisible by 3, and 737 is divisible by 11) and so is the only three-digit splendiferous number. A four-digit splendiferous number would have its first and last three digits be 373 at the same time, which is impossible; hence, there are no four-digit splendiferous numbers. There cannot be a splendiferous number with five or more digits since there are no four-digit splendiferous numbers. Hence, the sum of all splendiferous numbers is $2+3+5+7+23+37+53+73+373=576$.

27. $2\left(\frac{1}{2}\right)(1)(1)=1$

28. Let P and $P+100$ be the two primes, and let R be their concatenation. Consider P modulo 3. If $P\equiv 1 \bmod 3$, then $P+100\equiv 2 \bmod 3$. Since the remainder upon division by 3 is given by the sum of the digits, and R is made up of P and $P+100$, then R has a digit sum congruent to $1+2=3$ modulo 3, which is divisible by 3, and thus is itself divisible by 3. Hence R cannot be prime if $P\equiv 1$ mod 3. Now consider $P\equiv 2 \bmod 3$. Then $P+100\equiv 0 \bmod 3$, so it cannot be prime. Therefore $P\equiv 0 \bmod 3$, but the only prime divisible by 3 is 3. Hence the two primes are 103 and 3, and their concatenation is 1033, which is prime. The sum of the digits is $1+0+3+3=7$.

29. Let $f(x)=1+x+x^2+x^3+\cdots+x^{200}$. Then it is easy to see that
$$x\cdot f'(x)=x+2x^2+3x^3+4x^4+\cdots+200x^{200}.$$
Hence, the problem is asking us to compute $2\cdot f'(2)$. Since $f(x)$ is a finite geometric series, we have, for $x\neq 1$,
$$f(x)=\frac{x^{201}-1}{x-1}.$$
Then,
$$xf'(x)=x\left(\frac{201x^{200}(x-1)-(x^{201}-1)}{(x-1)^2}\right)=\frac{200x^{202}-201x^{201}+x}{(x-1)^2}$$
and for $x=2$ we have
$$2f'(2)=\frac{200\cdot 2^{202}-201\cdot 2^{201}+2}{1}=(400-201)2^{201}+2=199\cdot 2^{201}+2.$$

30. To obtain the minimum possible average with distinct integers, we must have the set S as
$$S=\{0,1,2,3,4,\dots,k,2079\}$$
for some integer k satisfying $0\leq k\leq 2078$. The average of the numbers in the set is therefore
$$\frac{1+2+3+\cdots+k+2079}{k+2}=\frac{\frac{k(k+1)}{2}+2079}{k+2}.$$
Manipulating this expression yields
$$\frac{1}{2}\left(\frac{k^2+k-2+4160}{k+2}\right)=\frac{1}{2}\left(k-1+\frac{4160}{k+2}+3-3\right)=\frac{1}{2}\left(k+2+\frac{4160}{k+2}\right)-\frac{3}{2}.$$
The expression in parentheses must satisfy
$$k+2+\frac{4160}{k+2}\geq 2\sqrt{4160}$$
because $\left(\sqrt{k+2}-\sqrt{\frac{4160}{k+2}}\right)^2\geq 0$. Equality occurs if and only if $k+2=\sqrt{4160}$, which is between 64 and 65. Hence, k is between 62 and 63. Since $k+2+\frac{4160}{k+2}$ is a convex function for $0\leq k\leq 2078$, the

minimum possible value of the average must occur when $k = 62$ or $k = 63$. If $k = 62$, the average of the integers in S is $\frac{1}{2}\left(64 + \frac{4160}{64}\right) - \frac{3}{2} = 63$. If $k = 63$, the average of the integers in S is $\frac{1}{2}\left(65 + \frac{4160}{65}\right) - \frac{3}{2} = 63$. In either case, the minimum is 63.