1) Find $f'(3)$ if $f(x) = x\cos(2x)$

(A) $\cos(3) + 3\sin(3)$ (B) $\cos(6) - 3\sin(6)$

(C) $\cos(6) - 6\sin(6)$ (D) $\cos(6) + 6\sin(6)$ (E) NOTA

Solution: $f'(x) = \cos(2x) - 2x\sin(2x) \to f'(3) = \cos(6) - 6\sin(6)$. C.

2) Find the slope of the line normal to $f(x) = e^{2x}$ at $x = \ln(2)$.

(A) 4 (B) $-\frac{1}{4}$

(C) 8 (D) $-\frac{1}{8}$ (E) NOTA

Solution: $f'(x) = 2e^{2x} \to f'(\ln(2)) = 2e^{2\ln(2)} = 8$. So the slope of the normal line is $-\frac{1}{8}$. D.

3) Find $\frac{d}{dx}\left[\frac{\sin(\pi x)+1}{x^2+4}\right]\Big|_{x=1}$

(A) $\frac{\pi}{5} + \frac{2}{25}$ (B) $-\frac{\pi}{5} + \frac{2}{25}$

(C) $\frac{\pi}{5} - \frac{2}{25}$ (D) $-\frac{\pi}{5} - \frac{2}{25}$ (E) NOTA

Solution: $\frac{d}{dx}\left[\frac{\sin(\pi x)+1}{x^2+4}\right]\Big|_{x=1} = \left[\frac{(x^2+4)(\pi\cos(\pi x)) - 2x(\sin(\pi x)+1)}{(x^2+4)^2}\right]\Big|_{x=1} = \frac{-5\pi-2}{25} = -\frac{\pi}{5} - \frac{2}{25}$. D.

4) Find $\lim_{x\to 3}\frac{x^3-3x^2-x+3}{x^3+x^2-9x-9}$.

(A) $\frac{1}{3}$ (B) $-\frac{1}{3}$

(C) $-\frac{2}{3}$ (D) $\frac{2}{3}$ (E) NOTA

Solution: $\lim_{x\to 3}\frac{x^3-3x^2-x+3}{x^3+x^2-9x-9} = \lim_{x\to 3}\frac{x(x^2-1)-3(x^2-1)}{x^2(x+1)-9(x+1)} = \lim_{x\to 3}\frac{(x-3)(x+1)(x-1)}{(x-3)(x+3)(x+1)} = \lim_{x\to 3}\frac{(x-1)}{(x+3)} = \frac{1}{3}$. A.

5) Find the equation of the tangent line to the curve $x^3 - xy^2 + 2y^4 = 8$ at the point $(2,1)$.

(A) $y = -\frac{11}{4}x + \frac{9}{2}$ (B) $y = \frac{11}{4}x - \frac{9}{2}$

(C) $y = -\frac{11}{4}x + \frac{13}{2}$ (D) $y = \frac{11}{4}x - \frac{13}{2}$ (E) NOTA

Solution: $x^3 - xy^2 + 2y^4 = 8 \to 3x^2 - y^2 - 2xy\frac{dy}{dx} + 8y^3\frac{dy}{dx} = 0 \to \frac{dy}{dx} = \frac{-3x^2+y^2}{8y^3-2xy} = \frac{-3(4)+1}{8(1)-2(2)(1)} = -\frac{11}{4}$. So the equation of the tangent line is $y - 1 = -\frac{11}{4}(x-2) \to y = -\frac{11}{4}x + \frac{13}{2}$. C.

6) Approximate the area between the curve $y = x^3 + 1$ and the x-axis from $x = 1$ to $x = 3$ using the Trapezoidal Rule with four intervals of equal width.

(A) 16 (B) $\frac{45}{2}$

(C) 29 (D) 45 (E) NOTA

Solution: The easiest way is to average the Left- and Right-handed Riemann sums: $y(1) = 2, y\left(\frac{3}{2}\right) = \frac{35}{8}, y(2) = 9, y\left(\frac{5}{2}\right) = \frac{133}{8}$, and $y(3) = 28$. The width of each interval is $\frac{1}{2}$, so the final result is $\frac{1}{2}\left(\frac{1}{2}\left(2 + \frac{35}{8} + 9 + \frac{133}{8}\right) + \frac{1}{2}\left(\frac{35}{8} + 9 + \frac{133}{8} + 28\right)\right) = \frac{45}{2}$. B.

7) Evaluate: $\int_1^2 \left(x^3 + \frac{1}{x^2}\right) dx$

(A) $\frac{7}{2}$ (B) $\frac{3}{4}$

(C) $\frac{17}{4}$ (D) 2 (E) NOTA

Solution: $\int_1^2 \left(x^3 + \frac{1}{x^2}\right) dx = \left[\frac{1}{4}x^4 - \frac{1}{x}\right]_1^2 = 4 - \frac{1}{2} - \frac{1}{4} + 1 = \frac{17}{4}$. C.

8) Find $\int_{-3}^{3} \sqrt{9 - x^2} dx$

(A) 9π (B) $\frac{9}{2}\pi$

(C) $\frac{9}{4}\pi$ (D) $\frac{9}{8}\pi$ (E) NOTA

Solution: This is the area of half a circle of radius 3, which is $\frac{9}{2}\pi$. B.

9) Evaluate: $\int_0^1 \frac{x-1}{x^2-2x+5} dx$

(A) $\ln\left(\frac{2\sqrt{5}}{5}\right)$ (B) $\ln\left(\frac{4}{5}\right)$

(C) $\ln\left(\frac{\sqrt{5}}{2}\right)$ (D) $\ln\left(\frac{2\sqrt{5}}{2}\right)$ (E) NOTA

Solution: $\int_0^1 \frac{x-1}{x^2-2x+5} dx = \frac{1}{2}\int_5^4 \frac{du}{u} = \frac{1}{2}(\ln(4) - \ln(5)) = \ln\left(\sqrt{\frac{4}{5}}\right) = \ln\left(\frac{2\sqrt{5}}{5}\right)$. A.

10) Evaluate: $\int_1^3 \frac{1}{x^2-2x+5} dx$

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{4}$

(C) $\frac{\pi}{8}$ (D) $\frac{\pi}{16}$ (E) NOTA

Solution: $\int_1^3 \frac{1}{x^2-2x+5}dx = \int_1^3 \frac{1}{(x-1)^2+4}dx = \left[\frac{1}{2}\arctan\left(\frac{x-1}{2}\right)\right]_1^3 = \frac{\pi}{8}$. C.

11) Evaluate: $\int_0^1 \frac{1}{x^2-2x-3}dx$

(A) $\frac{\ln(3)}{3}$ (B) $\frac{\ln(3)}{4}$

(C) $-\frac{\ln(3)}{3}$ (D) $-\frac{\ln(3)}{4}$ (E) NOTA

Solution: $\int_0^1 \frac{1}{x^2-2x-3}dx = \int_0^1 \frac{1}{(x-3)(x+1)}dx = \int_0^1 \left(\frac{1/4}{x-3} - \frac{1/4}{x+1}\right)dx = \frac{1}{4}[\ln|x-3| - \ln|x+1|]_0^1 = \frac{1}{4}(\ln(2) - \ln(2) - \ln(3) + \ln(1)) = -\frac{\ln(3)}{4}$. D.

12) Find $\lim\limits_{n\to\infty}\left(\sum_{k=1}^{n}\frac{k}{k^2+n^2}\right)$.

(A) $\ln(2)$ (B) $\frac{1}{2}\ln(2)$

(C) $\ln\left(\frac{5}{2}\right)$ (D) $\frac{1}{2}\ln\left(\frac{5}{2}\right)$ (E) NOTA

Solution: $\lim\limits_{n\to\infty}\left(\sum_{k=1}^{n}\frac{k}{k^2+n^2}\right) = \lim\limits_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{n}\frac{\frac{k}{n}}{\left(\frac{k}{n}\right)^2+1}\right) = \int_0^1 \frac{x}{x^2+1}dx = \frac{1}{2}\ln(2)$. B.

13) Find $\lim\limits_{x\to\infty}\left(\sqrt{3x^2-2x+5} - \sqrt{3x^2-7x+11}\right)$

(A) $\frac{5\sqrt{3}}{6}$ (B) $\frac{5\sqrt{3}}{3}$

(C) $-\frac{5\sqrt{3}}{6}$ (D) $-\frac{5\sqrt{3}}{3}$ (E) NOTA

Solution: $\lim\limits_{x\to\infty}\left(\sqrt{3x^2-2x+5} - \sqrt{3x^2-7x+11}\right) = \lim\limits_{x\to\infty}\frac{(3x^2-2x+5)-(3x^2-7x+11)}{\sqrt{3x^2-2x+5}+\sqrt{3x^2-7x+11}} = \lim\limits_{x\to\infty}\frac{5x-6}{x\left(\sqrt{3-2/x+5/x^2}+\sqrt{3-7/x+11/x^2}\right)} = \frac{5}{2\sqrt{3}} = \frac{5\sqrt{3}}{6}$. A.

14) Find $\frac{d}{dx}[x^x]$.

(A) x^x (B) $x^x\ln(x)$

(C) $x^x(\ln(x)+1)$ (D) $x^x(\ln(x)-1)$ (E) NOTA

Solution: $x^x = y \to x\ln(x) = \ln(y) \to \ln(x) + 1 = \frac{y'}{y} \to y' = x^x(\ln(x)+1)$. C.

15) The functions $f(x) = x^2 + 1$ and $g(x) = -x^2$ share a common tangent line of positive slope. What is its equation?

(A) $y = \frac{\sqrt{2}}{2}x + 1$ (B) $y = \frac{\sqrt{2}}{2}x + \frac{1}{2}$

(C) $y = \sqrt{2}x + 1$ (D) $y = \sqrt{2}x + \frac{1}{2}$ (E) NOTA

Solution: We are looking for points $(x_1, {x_1}^2 + 1)$ and $(x_2, -{x_2}^2)$ such that $2x_1 = -2x_2 = \frac{{x_1}^2+1+{x_2}^2}{x_1-x_2}$. The first equality implies $x_1 = -x_2$ so plugging that into the final term yields $2x_1 = \frac{2{x_1}^2+1}{2x_1} \to 2x_1^2 = 1 \to x_1 = \frac{\sqrt{2}}{2}$. Therefore the slope is $2x_1 = \sqrt{2}$ and one of the points is $\left(\frac{\sqrt{2}}{2}, \frac{3}{2}\right)$. So the equation of the tangent line is $y - \frac{3}{2} = \sqrt{2}\left(x - \frac{\sqrt{2}}{2}\right) \to y = \sqrt{2}x + \frac{1}{2}$. D.

16) Find the range of $f(x) = \frac{x}{x^6+1}$.

(A) $\left[-\frac{5^{\frac{1}{6}}}{6}, \frac{5^{\frac{1}{6}}}{6}\right]$ (B) $\left[-5^{\frac{1}{6}}, 5^{\frac{1}{6}}\right]$

(C) $\left[-\frac{5^{\frac{5}{6}}}{6}, \frac{5^{\frac{5}{6}}}{6}\right]$ (D) $\left[-5^{\frac{5}{6}}, 5^{\frac{5}{6}}\right]$ (E) NOTA

Solution: $f(x) = \frac{x}{x^6+1} \to f'(x) = \frac{(x^6+1)(1)-x(6x^5)}{(x^6+1)^2} = \frac{-5x^6+1}{(x^6+1)^2} \to x_{max} = \pm\sqrt[6]{\frac{1}{5}} \to f(x_{max}) = \pm\frac{5}{6}\sqrt[6]{\frac{1}{5}} = \pm\frac{5^{\frac{5}{6}}}{6} \to$ the range is $\left[-\frac{5^{\frac{5}{6}}}{6}, \frac{5^{\frac{5}{6}}}{6}\right]$. C.

17) Evaluate: $\int_0^{\sqrt{\pi}} e^x(\cos(x^2) - 2x\sin(x^2))dx$

(A) $-e^{\sqrt{\pi}} - 1$ (B) $e^{\sqrt{\pi}} - 1$

(C) $-e^{\sqrt{\pi}} + 1$ (D) $e^{\sqrt{\pi}} + 1$ (E) NOTA

Solution: $\int_0^{\sqrt{\pi}} e^x(\cos(x^2) - 2x\sin(x^2))dx = \int_0^{\sqrt{\pi}} \frac{d}{dx}[e^x\cos(x^2)]dx = -e^{\sqrt{\pi}} - 1$. A.

18) Evaluate: $\int_0^{\pi/3} \sec(x)\tan(x) \cdot \ln|\sec(x) + \tan(x)|\,dx$

(A) $2\ln|2 + \sqrt{3}| + \sqrt{3}$ (B) $2\ln|2 + \sqrt{3}|$

(C) $2\ln|2 + \sqrt{3}| - \sqrt{3}$ (D) $\sqrt{3}\ln|2 + \sqrt{3}|$ (E) NOTA

Solution: Use integration by parts with $u = \ln|\sec(x) + \tan(x)| \to du = \sec(x)\,dx$ and $dv = \sec(x)\tan(x)\,dx \to v = \sec(x)$. Then $\int_0^{\pi/3} \sec(x)\tan(x)\ln|\sec(x) + \tan(x)|\,dx = [\sec(x)\ln|\sec(x) + \tan(x)|]_0^{\pi/3} - \int_0^{\frac{\pi}{3}} \sec^2(x)\,dx = [\sec(x)\ln|\sec(x) + \tan(x)| - \tan(x)]_0^{\frac{\pi}{3}} = 2\ln|2 + \sqrt{3}| - \sqrt{3}$. C.

19) From the top of a tree 30 meters tall, a monkey is pulling up a bundle of bananas attached to a rope. The bundle of bananas has a weight of 20 Newtons, and the rope has a linear density of 6 Newtons per meter. How much work (in Newton-meters) does the monkey do when pulling the bundle of bananas to the top of the tree?

(A) 600 (B) 780

(C) 3,300 (D) 6,000 (E) NOTA

Solution: Let x be the height of the bananas from the ground. Then, to move a small height dx will result in an amount of work $dW = 20dx + 6xdx \rightarrow W = \int_0^{30}(20+6x)dx = [20x + 3x^2]_0^{30} = 600 + 2700 = 3300$. C.

20) For which of the following functions is the average rate of change of the function equal to the average value of the function over any real interval?

(A) $f(x) = 2019$ (B) $f(x) = \sin(x)$

(C) $f(x) = x^2$ (D) $f(x) = e^x$ (E) NOTA

Solution: $\frac{f(b)-f(a)}{b-a} = \frac{1}{b-a}\int_a^b f(x)dx \rightarrow f(b) - f(a) = F(b) - F(a)$ for any interval (a, b). This is clearly true for all intervals only if a function is its own antiderivative. D.

21) Consider the region between $f(x) = x^r$ $(r \geq 1)$ and the x-axis from $x = 0$ to $x = a$. If, for any real value of a, the y-coordinate of the centroid of this region is equal to the average value of $f(x)$ over the interval $(0, a)$, then what is r?

(A) $1-\sqrt{2}$ (B) 1

(C) $1+\sqrt{2}$ (D) 2 (E) NOTA

Solution: $\frac{1}{a}\int_0^a x^r dx = \frac{\int_0^a \frac{(x^r)^2}{2}dx}{\int_0^a x^r dx} \rightarrow \frac{1}{a}\left(\frac{1}{r+1}a^{r+1}\right) = \frac{1}{2}\frac{\frac{1}{2r+1}a^{2r+1}}{\frac{1}{r+1}a^{r+1}} \rightarrow \left(\frac{1}{r+1}\right)^2\frac{\left(a^{r+1}\right)^2}{a} = \frac{1}{4r+2}a^{2r+1} \rightarrow$ $(r+1)^2 = 4r+2 \rightarrow r^2+2r+1 = 4r+2 \rightarrow r^2-2r-1 = 0 \rightarrow r = 1 \pm \sqrt{2}$. Only $1+\sqrt{2} > 1$. C.

22) Find $\frac{d^{2019}}{dx^{2019}}[x^3\cos(x^2)]\Big|_{x=0}$

(A) $\frac{2019!}{1008!}$ (B) $\frac{2019!}{1009!}$

(C) $\frac{2018!}{1008!}$ (D) $\frac{2018!}{1009!}$ (E) NOTA

Solution: The Maclaurin series of $x^3\cos(x^2)$ is $x^3\sum_{n=0}^{\infty}\frac{(-1)^n(x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+3}}{(2n)!}$. In general, a Maclaurin series is of the form $\sum_{k=0}^{\infty}\frac{f^{(k)}(0)x^k}{k!}$. Therefore we can set $\frac{f^{(2019)}(0)x^{2019}}{2019!} = \frac{(-1)^n x^{4n+3}}{(2n)!}$ which

occurs when $4n+3=2019 \rightarrow n=504$. Therefore $\frac{f^{(2019)}(0)x^{2019}}{2019!}=\frac{(-1)^{504}x^{2019}}{1008!} \rightarrow f^{(2019)}(0)=\frac{2019!}{1008!}$.
A.

23) The finite region in the first quadrant bounded by the x- and y-axes and the curve $y=r^2-x^2$ is divided into two regions of equal area by the curve $y=ax^2$. Assume r is a non-zero real constant. Find a.

(A) 2 (B) 3

(C) 4 (D) 5 (E) NOTA

Solution: The total area of the region described is $\int_0^r (r^2-x^2)dx=\left[r^2x-\frac{1}{3}x^3\right]_0^r=\frac{2}{3}r^3$. The two curves intersect when $ax^2=r^2-x^2 \rightarrow (a+1)x^2=r^2 \rightarrow x=\frac{r}{\sqrt{a+1}}$. So we want to find a such that $\int_0^{\frac{r}{\sqrt{a+1}}}(r^2-x^2-ax^2)dx \rightarrow \int_0^{\frac{r}{\sqrt{a+1}}}(r^2-(a+1)x^2)dx=\left[r^2x-\frac{1}{3}(a+1)x^3\right]_0^{\frac{r}{\sqrt{a+1}}}=\frac{r^3}{\sqrt{a+1}}-\frac{1}{3}(a+1)\frac{r^3}{(a+1)^{\frac{3}{2}}}=\frac{2}{3}\frac{r^3}{\sqrt{a+1}}=\frac{1}{3}r^3 \rightarrow \sqrt{a+1}=2 \rightarrow a=3$. B.

24) Consider a matrix of the form $\begin{bmatrix} x & 1 & 0 \\ y^2 & y & 5 \\ x & 1 & y \end{bmatrix}$ with non-negative entries and a determinant of 12. What is the maximum possible trace of such a matrix?

(A) 1 (B) 3

(C) 6 (D) 9 (E) NOTA

Solution: $det\begin{bmatrix} x & 1 & 0 \\ y^2 & y & 5 \\ x & 1 & y \end{bmatrix}=xy^2+5x-5x-y^3=(x-y)y^2=12 \rightarrow x=\frac{12}{y^2}+y$. Let $T \equiv tr\begin{bmatrix} x & 1 & 0 \\ y^2 & y & 5 \\ x & 1 & y \end{bmatrix}=2y+x=2y+\frac{12}{y^2}+y=3y+\frac{12}{y^2} \rightarrow T'=3-\frac{24}{y^3}=0 \rightarrow y^3=8 \rightarrow y=2 \rightarrow T=9$.
D.

25) Evaluate for $k>0$: $\int_0^\infty \frac{dx}{\cosh(x)+k\sin\ (x)+1}$

(A) $\frac{1}{k}\ln(k)$ (B) $\frac{1}{k}\ln(k+1)$

(C) $\frac{1}{k+1}\ln(k)$ (D) $\frac{1}{k+1}\ln(k+1)$ (E) NOTA

Solution: There are multiple ways to do this integral, but one of the most direct is to use the equivalent of the tangent half-angle approximation $t=\tanh\left(\frac{x}{2}\right)$. Because $\tanh(i\theta)=i\tan(\theta)$, this results in the t^2 terms changing signs from the standard substitution: $\sinh(x)=\frac{2t}{1-t^2}$, $\cosh(x)=\frac{1+t^2}{1-t^2}$, and $dx=$

$\frac{2d}{1-t^2}$. So $\int_0^\infty \frac{dx}{\cosh(x)+k\sinh(x)+1} = \int_0^1 \frac{\frac{2dt}{1-t^2}}{\frac{1+t^2}{1-t^2}+k\frac{2t}{1-t^2}+1} = \int_0^1 \frac{2d}{1+t^2+2kt+1-t^2} = \int_0^1 \frac{dt}{kt+1} = \left[\frac{1}{k}\ln(kt+1)\right]_0^1 = \frac{1}{k}\ln(k+1)$. B.

26) Let $f(x)$ be a continuous, differentiable function such that, for any real $a \geq 0$, $\int_1^\infty e^{-ax} f(x)dx = \frac{a^2}{a^6+1}$. Find $\int_1^\infty \frac{f(x)}{x}dx$.

(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{4}$

(C) $\frac{\pi}{6}$ (D) $\frac{\pi}{12}$ (E) NOTA

Solution: Let $I(a) = \int_1^\infty e^{-a}\ \frac{f(x)}{x}dx$. Then $I'(a) = -\int_1^\infty e^{-ax}f(x)dx = -\frac{a^2}{a^6+1}$ based on the statement of the question. Therefore $I(a) = -\int \frac{a^2}{a^6+1}da = -\int \frac{a^2}{(a^3)^2+1}da = -\frac{1}{3}\int \frac{du}{u^2+1} = C - \frac{1}{3}\arctan(u) = C - \frac{1}{3}\arctan(a^3)$. Note also that $\lim_{a\to\infty} I(a) = \lim_{a\to\infty} \int_1^\infty e^{-a}\ \frac{f(x)}{x}dx = 0$. Therefore $0 = \lim_{a\to\infty}\left(C - \frac{1}{3}\arctan(a^3)\right) = C - \frac{\pi}{6} \to C = \frac{\pi}{6}$. Thus, $I(a) = \frac{\pi}{6} - \frac{1}{3}\arctan(a^3)$. The desired integral is $I(0) = \int_1^\infty \frac{f(x)}{x}dx = \frac{\pi}{6}$. C.

27) Which of the following is equivalent to $\frac{d}{d\left(\frac{d}{d\left(\frac{d}{d\left(\frac{d}{d(\because)}[x^{2019}]\right)}[x^{2019}]\right)}[x^{2019}]\right)}[x^{2019}]$?

(A) $2019x^{2018}$ (B) $2019x^{1009}$

(C) $x^{\frac{2019}{2}}$ (D) $\sqrt{2}x^{\frac{2019}{2}}$ (E) NOTA

Solution: Let $u(x) = \frac{d}{d\left(\frac{d}{d\left(\frac{d}{d(\because)}[x^{2019}]\right)}[x^{2019}]\right)}[x^{2019}]$. Then we have $\frac{d}{du}[x^{2019}] = u(x) \to 2019x^{2018}\frac{dx}{du} = u(x) \to 2019x^{2018} = u(x)\frac{du}{dx}$. Since $u(x) = 2019x^{2018}\frac{dx}{du}$ it is clear that $u(0) = 0$, so $2019x^{2018} = u(x)\frac{du}{dx} \to x^{2019} = \frac{1}{2}u^2 \to u(x) = \sqrt{2}x^{\frac{2019}{2}}$. D.

28) Let $f(x)$, $g(x)$, and $h(x)$ be continuously differentiable functions with the following properties:

$$f'(x) = f(x) + g(x) + h(x)$$
$$g'(x) = f(x) - 2h(x)$$
$$h'(x) = 2f(x) - g(x) + h(x)$$

Further, $f(0) = 0$, $g(0) = 3$, and $h(0) = -1$. Find $f(\ln(2))$.

(A) $\frac{15}{16}$ (B) $\frac{7}{8}$

(C) $\frac{15}{8}$ (D) $\frac{7}{4}$ (E) NOTA

Solution: $f'(x) = f(x) + g(x) + h(x) \to f''(x) = f'(x) + g'(x) + h'(x) = f(x) + g(x) + h(x) + f(x) - 2h(x) + 2f(x) - g(x) + h(x) = 4f(x)$. So $f''(x) = 4f(x) \to f(x) = C_1e^{2x} + C_2e^{-2x}$. $f(0) = C_1 + C_2 = 0$ and $f'(x) = 2C_1e^{2x} - 2C_2e^{-2x} \to f'(0) = 2C_1 - 2C_2 = 0 + 3 - 1 = 2$. The solution to $C_1 + C_2 = 0$ and $C_1 - C_2 = 1$ is $C_1 = \frac{1}{2}$ and $C_2 = -\frac{1}{2}$. So $f(x) = \frac{1}{2}e^{2x} - \frac{1}{2}e^{-2x} \to f(\ln(2)) = \frac{1}{2}e^{2\ln(2)} - \frac{1}{2}e^{-2\ln(2)} = \frac{1}{2}(4) - \frac{1}{2}\left(\frac{1}{4}\right) = 2 - \frac{1}{8} = \frac{15}{8}$. C.

29) The point P begins at the origin and moves in the Cartesian plane along the line $y = \frac{1}{2}x$ such that the x-cordinate of P is changing at a rate of $+3$ units per second. Consider the area enclosed by the locus of all points that are exactly $\frac{1}{3}$ as far away from the point P as they are from the line $y = -2x$. At what rate is this area changing when $P = (8, 4)$?

(A) $\frac{81\pi\sqrt{2}}{2}$ (B) $\frac{81\pi\sqrt{2}}{4}$

(C) $\frac{9\pi\sqrt{2}}{2}$ (D) $\frac{9\pi\sqrt{2}}{4}$ (E) NOTA

Solution: The locus describes an ellipse with eccentricity $\frac{1}{3}$ with focus $P = (p, q)$ and directrix $y = -2x$. The vertex will be along the line $y = \frac{1}{2}x$, which intersects the directrix at the origin. In general, the point that is $\frac{2}{3}$ the distance from $(0,0)$ to (p, q) is $\left(\frac{3p}{4}, \frac{3q}{4}\right)$. This is the location of one of the vertices. The distance from (p, q) to $\left(\frac{3p}{4}, \frac{3q}{4}\right)$ is $\frac{\sqrt{p^2+q^2}}{4} = a - c$ where a is the length of the semi-major axis and $c = \sqrt{a^2 - b^2}$ is the focal distance, with b the length of the semi-minor axis. We also know that $e = \frac{c}{a} = \frac{1}{3}$. Therefore $c = \frac{1}{3}a \to \frac{\sqrt{p^2+q^2}}{4} = a - \frac{1}{3}a = \frac{2}{3}a \to a = \frac{3\sqrt{p^2+q^2}}{8}$ and $c^2 = a^2 - b^2 \to \frac{1}{9}a^2 = a^2 - b^2 \to b^2 = \frac{8}{9}a^2 \to b = \frac{2\sqrt{2}}{3}a = \frac{2\sqrt{2}}{3} \cdot \frac{3\sqrt{p^2+q^2}}{8} = \frac{\sqrt{2}}{4}\sqrt{p^2 + q^2}$. Thus the area is $A = \pi ab = \pi \frac{3\sqrt{p^2+q^2}}{8} \cdot \frac{\sqrt{2}}{4}\sqrt{p^2 + q^2} = \frac{3\pi\sqrt{2}}{32}(p^2 + q^2)$. Thus $\frac{dA}{dt} = \frac{3\pi\sqrt{2}}{16}\left(p\frac{dp}{dt} + q\frac{dq}{dt}\right)$. Since $\frac{dp}{dt} = 3$ and $q = \frac{1}{2}p$ then $\frac{dq}{dt} = \frac{3}{2}$ and therefore $\frac{dA}{dt} = \frac{3\pi\sqrt{2}}{16}\left(3p + \frac{3}{2}q\right) = \frac{3\pi\sqrt{2}}{16}(24 + 6) = \frac{45\pi\sqrt{2}}{8}$. E.

30) Find $\lim_{n\to\infty}\left(1 + \frac{3}{n}\right)^{5n}$

(A) e^{15} (B) e^5

(C) e^3 (D) $e^{5/3}$ (E) NOTA

Solution: $\lim_{n\to\infty}\left(1 + \frac{r}{n}\right)^{nt} = e^{rt}$ so $\lim_{n\to\infty}\left(1 + \frac{3}{n}\right)^{5n} = e^{15}$. A.