1. C
2. C
3. E
4. A
5. D
6. C
7. C
8. B
9. A
10. A
11. B
12. C
13. E
14. B
15. B
16. D
17. E
18. C
19. C
20. C
21. E
22. B
23. B
24. E
25. C
26. A
27. B
28. A
29. C
30. D

1. Using the fact that if $a|b$ and $a|c$ then $a|jb+kc$, can check each of the answer choices to see which one is divisible by 5. Assuming that $5|3x+y$, we know that $5|3(x+4y)-3x+y \rightarrow 5|11y$ which is a contradiction. A similar process shows that $5 \nmid 7x+6y, 2x+5y$. We know that $5|6x+9y$ since $5|6(x+4y)-5(3y)$ $\boxed{C}$

2. 17^{-1} is equivalent to x if x satisfies the equivalence $17x \equiv 1 \mod 47$. Then by definition of mod, there exists n_1 such that $47n_1 = 17x - 1$. Taking $\mod 17$ of each side gives the equivalence $13n_1 \equiv -1 \mod 17$. Then $17n_2 = 13n_1 + 1$ and taking $\mod 13$ gives $4n_2 \equiv 1 \mod 13$. Then $13n_3 = 4n_2 - 1$ and taking $\mod 4$ gives $n_3 \equiv -1 \mod 4$. Thus $n_3 \equiv 3 \mod 4$ which gives $n_2 \equiv 10 \mod 13$ which gives $n_1 \equiv 13 \mod 17$ and finally $x \equiv 36 \mod 47$ $\boxed{C}$

3. This is famously known as a "taxicab number", the smallest of which is 1729 $\boxed{E}$

4. The expansion of AB_{10} is $10a + b$ and similarly the expansion of BA_7 is $7b + a$. Then $10a+b = 7b+a \rightarrow 3a = 2b$. Then since a and b are digits in base 7, the only possibilities for (a, b) are $(2, 3)$ and $(4, 6)$. Then the sum of AB_{10} is $23 + 46 = 69$ $\boxed{A}$

5. If x is the number of Pokemon Ben has caught so far then

$$x \equiv 5 \mod 16$$
$$x \equiv 4 \mod 9$$
$$x \equiv 3 \mod 5$$

Combining the first two equivalences gives the equation $16a + 5 = 9b + 4$. Noticing that $a = 5$, $b = 9$ is a solution (or using Euclidean algorithm), this gives $x \equiv 85 \mod 144$. By the same process we combine mod144 and mod5, which gives $x \equiv 373 \mod 720$. Since there are only 802 Pokemon available, Ben must have caught 373 Pokemon $\boxed{D}$

6. After n dice rolls, the sum has equal probabilities of being 0, 1, ..., 5 $\mod 6$. After that roll there is exactly one roll that will make the sum $0 \mod 6$. Thus the probability is $\frac{1}{6}$ $\boxed{C}$

7. $\phi(60) = 16$ (Euler's totient function) $\boxed{C}$

8. The Chicken McNugget Theorem states that the largest integer that cannot be represented by $ax + by$ for non-negative integers x, y and relatively prime a, b is $ab - a - b$. So $c = 6 \cdot 7 - 6 - 7 = 29$. The sum of digits in c is $2 + 9 = 11$ $\boxed{B}$

9. $x^2+2x+32 \equiv x^2+2x-3 \equiv (x+3)(x-1) \equiv 0 \bmod 35$ This implies that we must have

$$(x+3)(x-1) \equiv 0 \bmod 5$$
$$(x+3)(x-1) \equiv 0 \bmod 7$$

Clearly the first equivalence gives the solutions $x \equiv 1,2 \bmod 5$ and the second gives the solutions $x \equiv 1,4 \bmod 7$. Combining the possibilities gives $x \equiv 1,11,22,32 \bmod 35$. Then the sum is $1+11+22+32 \equiv 31 \bmod 35$ $\boxed{A}$

10. By principle of inclusion-exclusion, the number of integers is $\lfloor \frac{1500}{7} \rfloor + \lfloor \frac{1500}{11} \rfloor - 2\lfloor \frac{1500}{77} \rfloor = 312$ $\boxed{A}$

11. An integer that has 9 factors must either be in the form $p_1^2 p_2^2$ or p^8. The three smallest integers in this form are $2^2 3^2$, $2^2 5^2$, and $2^2 7^2$ $\boxed{B}$

12. The number of zeros at the end of 2018! are limited by the power of 5 that divides into 2018!. Counting powers of 5 yields $\lfloor \frac{2018}{5} \rfloor + \lfloor \frac{2018}{25} \rfloor + \lfloor \frac{2018}{125} \rfloor + \lfloor \frac{2018}{625} \rfloor = 502$ $\boxed{C}$

13. Since $gcd(a,b) \cdot lcm(a,b) = ab$, then our product is $84 \cdot 126 = 10584$ $\boxed{E}$

14. Counting the pairs (m,n) gives $(2,19),(7,17),\ldots,(47,1)$ which is 10 pairs in total $\boxed{B}$

15. The prime factorization of 3288 is $3288 = 2^3 \cdot 3 \cdot 137$. Then the sum of the factors of 3288 is $(2^0+2^1+2^2+2^3)(3^0+3^1)(137^0+137^1) = (15)(4)(138) = 8280$ $\boxed{B}$

16. Notice that the smallest possible product abc results from $a=3$, $b=4$, and $c=5$, which gives $abc=60$. We verify that the prime factors of 60 will always divide abc. For 3, notice that the quadratic residues modulo 3 are 0 and 1. Using casework we see that we can only have

$$a^2 \equiv 1 \pmod 3, b^2 \equiv 0 \pmod 3, c^2 \equiv 1 \pmod 3$$
$$a^2 \equiv 0 \pmod 3, b^2 \equiv 1 \pmod 3, c^2 \equiv 1 \pmod 3$$
$$a^2 \equiv 0 \pmod 3, b^2 \equiv 0 \pmod 3, c^2 \equiv 0 \pmod 3$$

In each case we can conclude that $3|abc$. Using similar logic for 4 and 5, we can show that $4|abc$ and $5|abc$. Thus the largest such k such that $k|abc$ is $k=60$ $\boxed{D}$

17. Examining the equation modulo 8 we see that $n^2 \equiv 3 \pmod 8$. Since 3 is not a quadratic residue of 8, there are no solutions to the equation. $\boxed{E}$

18. Since norm is multiplicative (this can be easily verified), $N((5+3i)(6-2i)(2+i)) = N(5+3i)N(6-2i)N(2+i) = (34)(40)(5) = 6800$ $\boxed{C}$

19. Since there exists γ such that $\gamma(1-5i) = \alpha$, we know that $N(\gamma)N(1-5i) = N(\alpha) \rightarrow \frac{N(\alpha)}{N(1-5i)} = N(\gamma) \in \mathbb{Z}$, which implies that $26|N(\alpha)$ $\boxed{C}$

20. Notice that $1, -1, i, -i$ are units. We eliminate answers by factoring.

$$2 = (1+i)(1-i)$$
$$41 = (5-4i)(5+4i)$$
$$1+5i = (1+i)(3+2i)$$

We must verify that $4+i$ is a prime. Assume for the sake of contradiction that $4+i = \alpha\beta$ for some non-units α, β. Then taking the norm of both sides we have $17 = N(\alpha)N(\beta)$. Then either $N(\alpha) = 1$ or $N(\beta) = 1$. This is a contradiction since neither α or β are units $\boxed{C}$

21. (I) is true since $N((a+bi)(c+di)) = N((ac-bd)+(ad+bc)i) = (ac-bd)^2+(ad+bc)^2 = (ac)^2+(bd)^2+(ad)^2+(bc)^2 = (a^2+b^2)(c^2+d^2) = N(a+bi)N(c+di)$.
(II) is false since the units of $\mathbb{Z}[i]$ are $1, -1, i, -i$.
(III) is false, however the converse is clearly true. As a counterexample, $N(1+2i)|N(1-2i)$ but $1+2i \nmid 1-2i$.
(IV) is true. Suppose there exists non-units $\alpha = a+bi$ and $\beta = c+di$ such that $p = \alpha\beta$. Then we have $N(p) = p^2 = N(\alpha)N(\beta)$. Since neither α nor β are units, we must have $a^2+b^2 = N(\alpha) = p = 4k+3$. But this is clearly impossible since a sum of squares cannot be 3 mod 4 $\boxed{E}$

22. Since $n^3+7n^2-13n+19 = (n-4)n^2+(n-4)11n+(n-4)31+143$ (synthetic division), then $n-4|n^3+7n^2-13n+19 \rightarrow n-4|143$. Then $n-4 = -1, 1, 11, 13, 143 \rightarrow n = 3, 5, 15, 17, 147$. The sum of all possible values of n is $3+5+15+17+147 = 187$ $\boxed{B}$

23. Using Euler's Totient Theorem, we know that $3^{72} \equiv 1 \bmod 91$. Then $3^{391} \equiv (3^{72})^5 3^{31} \equiv 3^{31} \bmod 91$. Then notice that,

$$3 \equiv 3 \bmod 91$$
$$3^2 \equiv 9 \bmod 91$$
$$3^4 \equiv -10 \bmod 91$$
$$3^8 \equiv 9 \bmod 91$$
$$3^{16} \equiv -10 \bmod 91$$
$$3^{32} \equiv 9 \bmod 91$$

$3^{32} \equiv 9 \bmod 91$ implies that $3^{31} \equiv 3 \bmod 91$ $\boxed{B}$

24. From Vieta's formula we know that $p_1 + p_2 + p_3 = 40$. If all the primes were odd then $p_1 + p_2 + p_3$ would be odd. Thus one of the primes must be 2. WLOG, let $p_1 = 2$. Then $p_2 + p_3 = 38$. Counting all the possible unordered triplets of (p_1, p_2, p_3) gives $(2, 7, 31)$ and $(2, 19, 19)$. Then the sum of all possible values for $|c_2|$ is $2 \cdot 7 \cdot 31 + 2 \cdot 19 \cdot 19 = 1156$ $\boxed{E}$

25. $\lim_{n\to\infty} (\frac{a_{n+1}}{a_n} = \frac{a_n}{a_{n-1}}) \rightarrow$

$$\begin{aligned}
\frac{3a_n - a_{n-1}}{a_n} &= \frac{a_n}{a_{n-1}} \\
a_{n-1}(3a_n - a_{n-1}) &= a_n^2 \\
a_n^2 - 3a_n a_{n-1} + a_{n-1}^2 &= 0 \\
(\frac{a_n}{a_{n-1}})^2 - 3\frac{a_n}{a_{n-1}} + 1 &= 0 \\
\frac{a_n}{a_{n-1}} &= \frac{3 \pm \sqrt{5}}{2}
\end{aligned}$$

By inspection the ratio is clearly greater than 1 so the ratio must be $\frac{3+\sqrt{5}}{2}$ $\boxed{C}$

26. We can examine the equation by various mods to determine possible values for x. Taking mod 2 we see that $1^5 + 0^5 + 0^5 + x^5 \equiv 0^5 \bmod 2 \rightarrow x \equiv 1 \bmod 2$. Taking mod 3 we see that $0^5 + 0^5 + 2^5 + x^5 \equiv 0^5 \bmod 3 \rightarrow x \equiv 1 \bmod 3$. Taking mod 5 we see that $2^5 + (-1)^5 + 0^5 + x^5 \equiv (-1)^2 \bmod 5 \rightarrow x \equiv 3 \bmod 5$. Taking mod 7 we see that $(-1)^5 + 0^5 + (-2)^5 + x^5 \equiv 4^5 \bmod 7 \rightarrow x \equiv 0 \bmod 7$. These conditions are enough to narrow the answer down to $x \equiv 133 \bmod 210$. We can computationally verify that 133 satisfies this equation $\boxed{A}$

27. Using Sophie-Germaine factorization,

$$\begin{aligned}
4^5 + 5^4 &= 5^4 + 4 \cdot 4^4 \\
&= (5^2 + 2 \cdot 5 \cdot 4 + 2 \cdot 4^2)(5^2 - 2 \cdot 5 \cdot 4 + 2 \cdot 4^2) \\
&= (25 + 40 + 32)(25 - 40 + 32) \\
&= (97)(17)
\end{aligned}$$

Thus there are 2 prime factors of $5^4 + 4^5$ $\boxed{B}$

28. First note that $3 \nmid x$. Next we add 1 to both sides and factor, which gives

$$(x+1)(x^2-x+1) = 3^y$$

Let $x+1 = 3^a$ and $x^2-x+1 = 3^b$ where $a+b = y$ and $b > a$. Then clearly $3^a | gcd(x-1, x^2+x+1)$ which is equivalent to $3^a | gcd(x-1, 3x)$ since $(x+1)^2-(x^2-x+1) = 3x$. Then clearly we must also have $3^a | 3x$, but since $3 \nmid x$, we can only have $a = 0, 1$. These give the only solutions $(0,0), (2,2)$. Thus the sum is $0+0+2+2 = 4$ $\boxed{A}$

29. Since $ab = k^2$ we can say $a = n_1^2 n_2$ and $b = n_2 n_3^2$. But $a-b = n_1^2 n_2 - n_2 n_3^2 = n_2(n_1^2 - n_3^2) = p$. Thus n_2 must be 1. Then $n_1^2 - n_3^2 = (n_1-n_3)(n_1+n_3) = p$. So $n_1 - n_3 = 1$ and $n_1 + n_3 = p$. Then our possible pairs (a,b) are $(4,1), (9,4), (16,9), (36,25), (49,36)$. The sum of all such a is $4+9+16+36+49 = 114$ $\boxed{C}$

30. Expanding gives

$$\begin{aligned} n^2-4n+4-m^2+4m-4 &= 2mn \\ n^2-4n-2mn-m^2+4m &= 0 \\ n^2+n(-4-2m)-m^2+4m &= 0 \end{aligned}$$

Considering this as a quadratic equation in n gives

$$\begin{aligned} n &= \frac{4+2m \pm \sqrt{(-4-2m)^2-4(-m^2+4m)}}{2} \\ &= \frac{4+2m \pm \sqrt{4m^2+16m+16+4m^2-16m}}{2} \\ &= \frac{4+2m \pm 2\sqrt{2m^2+4}}{2} \end{aligned}$$

Now since n must be an integer, this means that $\sqrt{2m^2+4}$ must be an integer. Then,

$$\sqrt{2m^2+4} = k \rightarrow k^2 - 2m^2 = 4$$

Notice that k must be even so using the substitution $k = 2k_1$ gives $2k_1^2 - m^2 = 2$. Then by the same logic m must also be even so using the substitution $m = 2m_1$ gives $k_1^2 - 2m_1^2 = 1$. This is a Pell's equation with base solution $(3,2)$, which gives the solution $(m,n) = (4,12)$. The next smallest solution (k_1, m_1) is given by

$$\begin{aligned} 3^2 - 2 \cdot 2^2 &= 1 \\ (3-2\sqrt{2})(3+2\sqrt{2}) &= 1 \\ (3-2\sqrt{2})^2(3+2\sqrt{2})^2 &= 1^2 \\ (17-12\sqrt{2})(17+12\sqrt{2}) &= 1 \end{aligned}$$

Thus $(k_1, m_1) = (17,12) \rightarrow (m,n) = (24,60)$. Then, $m+n = 24+60 = 84$ $\boxed{D}$