1. C
2. C
3. E
4. A
5. D
6. C
7. C
8. B
9. A
10. A
11. B
12. C
13. E
14. B
15. B
16. D
17. E
18. C
19. C
20. C
21. E
22. B
23. B
24. E
25. C
26. A
27. B
28. A
29. C
30. D
1. Using the fact that if \( a \mid b \) and \( a \mid c \) then \( a \mid b + kc \), can check each of the answer choices to see which one is divisible by 5. Assuming that \( 5 \mid 3x + y \), we know that \( 5 \mid 3(x + 4y) - 3x + y \rightarrow 5 \mid 11y \) which is a contradiction. A similar process shows that \( 5 \nmid 7x + 6y, 2x + 5y \). We know that \( 5 \mid 6x + 9y \) since \( 5 \mid 6(x + 4y) - 5(3y) \) \(\boxed{C}\)

2. \( 17^{-1} \) is equivalent to \( x \) if \( x \) satisfies the equivalence \( 17x \equiv 1 \mod 47 \). Then by definition of \( \mod \), there exists \( n \) such that \( 47n_1 = 17x - 1 \). Taking \( \mod 17 \) of each side gives the equivalence \( 13n_1 \equiv -1 \mod 17 \). Then \( 17n_2 = 13n_1 + 1 \) and taking \( \mod 13 \) gives \( 4n_2 \equiv 1 \mod 13 \). Then \( 13n_3 = 4n_2 - 1 \) and taking \( \mod 4 \) gives \( n_3 \equiv -1 \mod 4 \). Thus \( n_3 \equiv 3 \mod 4 \) which gives \( n_2 \equiv 10 \mod 13 \) which gives \( n_1 \equiv 13 \mod 17 \) and finally \( x \equiv 36 \mod 47 \) \(\boxed{C}\)

3. This is famously known as a "taxicab number", the smallest of which is 1729 \(\boxed{E}\)

4. The expansion of \( AB_{10} \) is \( 10a + b \) and similarly the expansion of \( BA_7 \) is \( 7b + a \). Then \( 10a + b = 7b + a \rightarrow 3a = 2b \). Then since \( a \) and \( b \) are digits in base 7, the only possibilities for \( (a, b) \) are \( (2, 3) \) and \( (4, 6) \). Then the sum of \( AB_{10} \) is \( 23 + 46 = 69 \) \(\boxed{A}\)

5. If \( x \) is the number of Pokemon Ben has caught so far then

\[
\begin{align*}
x & \equiv 5 \mod 16 \\
x & \equiv 4 \mod 9 \\
x & \equiv 3 \mod 5
\end{align*}
\]

Combining the first two equivalences gives the equation \( 16a + 5 = 9b + 4 \). Noticing that \( a = 5, b = 9 \) is a solution (or using Euclidean algorithm), this gives \( x \equiv 85 \mod 144 \). By the same process we combine \( \mod 144 \) and \( \mod 5 \), which gives \( x \equiv 373 \mod 720 \). Since there are only 802 Pokemon available, Ben must have caught 373 Pokemon \(\boxed{D}\)

6. After \( n \) dice rolls, the sum has equal probabilities of being \( 0, 1, \ldots, 5 \mod 6 \). After that roll there is exactly one roll that will make the sum 0 mod 6. Thus the probability is \( \frac{1}{6} \) \(\boxed{C}\)

7. \( \phi(60) = 16 \) (Euler’s totient function) \(\boxed{C}\)

8. The Chicken McNugget Theorem states that the largest integer that cannot be represented by \( ax + by \) for non-negative integers \( x, y \) and relatively prime \( a, b \) is \( ab - a - b \). So \( c = 6 \cdot 7 - 6 - 7 = 29 \). The sum of digits in \( c \) is \( 2 + 9 = 11 \) \(\boxed{B}\)
9. \( x^2 + 2x + 32 \equiv x^2 + 2x - 3 \equiv (x + 3)(x - 1) \equiv 0 \mod 35 \) This implies that we must have

\[
\begin{align*}
(x + 3)(x - 1) &\equiv 0 \mod 5 \\
(x + 3)(x - 1) &\equiv 0 \mod 7
\end{align*}
\]

Clearly the first equivalence gives the solutions \( x \equiv 1, 2 \mod 5 \) and the second gives the solutions \( x \equiv 1, 4 \mod 7 \). Combining the possibilities gives \( x \equiv 1, 11, 22, 32 \mod 35 \). Then the sum is \( 1 + 11 + 22 + 32 \equiv 31 \mod 35 \) [A]

10. By principle of inclusion-exclusion, the number of integers is \( \left\lfloor \frac{1500}{7} \right\rfloor + \left\lfloor \frac{1500}{11} \right\rfloor - 2\left\lfloor \frac{1500}{77} \right\rfloor = 312 \) [A]

11. An integer that has 9 factors must either be in the form \( p^2p_2^2 \) or \( p^8 \). The three smallest integers in this form are \( 2^23^2 \), \( 2^25^2 \), and \( 2^27^2 \) [B]

12. The number of zeros at the end of 2018! are limited by the power of 5 that divides into 2018!. Counting powers of 5 yields \( \left\lfloor \frac{2018}{5} \right\rfloor + \left\lfloor \frac{2018}{25} \right\rfloor + \left\lfloor \frac{2018}{125} \right\rfloor + \left\lfloor \frac{2018}{625} \right\rfloor = 502 \) [C]

13. Since \( \gcd(a, b) \cdot \text{lcm}(a, b) = ab \), then our product is \( 84 \cdot 126 = 10584 \) [E]

14. Counting the pairs \((m, n)\) gives \((2, 19), (7, 17), \ldots, (47, 1)\) which is 10 pairs in total [B]

15. The prime factorization of 3288 is \( 3288 = 2^3 \cdot 3 \cdot 137 \). Then the sum of the factors of 3288 is \((2^0 + 2^1 + 2^2 + 2^3)(3^0 + 3^1)(137^0 + 137^1) = (15)(4)(138) = 8280 \) [B]

16. Notice that the smallest possible product \( abc \) results from \( a = 3, b = 4, \) and \( c = 5 \), which gives \( abc = 60 \). We verify that the prime factors of 60 will always divide \( abc \). For 3, notice that the quadratic residues modulo 3 are 0 and 1. Using casework we see that we can only have

\[
\begin{align*}
a^2 &\equiv 1 \pmod{3}, b^2 \equiv 0 \pmod{3}, c^2 \equiv 1 \pmod{3} \\
a^2 &\equiv 0 \pmod{3}, b^2 \equiv 1 \pmod{3}, c^2 \equiv 1 \pmod{3} \\
a^2 &\equiv 0 \pmod{3}, b^2 \equiv 0 \pmod{3}, c^2 \equiv 0 \pmod{3}
\end{align*}
\]

In each case we can conclude that \( 3|abc \). Using similar logic for 4 and 5, we can show that \( 4|abc \) and \( 5|abc \). Thus the largest such \( k \) such that \( k|abc \) is \( k = 60 \) [D]

17. Examining the equation modulo 8 we see that \( n^2 \equiv 3 \pmod{8} \). Since 3 is not a quadratic residue of 8, there are no solutions to the equation [E]
18. Since norm is multiplicative (this can be easily verified), \( N((5 + 3i)(6 - 2i)(2 + i)) = N(5 + 3i)N(6 - 2i)N(2 + i) = (34)(40)(5) = 6800 \] 

19. Since there exists \( \gamma \) such that \( \gamma(1 - 5i) = \alpha \), we know that \( N(\gamma)N(1 - 5i) = N(\alpha) \to \frac{N(\alpha)}{N(1 - 5\alpha)} = N(\gamma) \in \mathbb{Z} \), which implies that \( 26|N(\alpha) \]

20. Notice that \( 1, -1, i, -i \) are units. We eliminate answers by factoring.

\[
\begin{align*}
2 &= (1 + i)(1 - i) \\
41 &= (5 - 4i)(5 + 4i) \\
1 + 5i &= (1 + i)(3 + 2i)
\end{align*}
\]

We must verify that \( 4 + i \) is a prime. Assume for the sake of contradiction that \( 4 + i = \alpha \beta \) for some non-units \( \alpha, \beta \). Then taking the norm of both sides we have \( 17 = N(\alpha)N(\beta) \). Then either \( N(\alpha) = 1 \) or \( N(\beta) = 1 \). This is a contradiction since neither \( \alpha \) or \( \beta \) are units

21. (I) is true since \( N((a + bi)(c + di)) = N((ac - bd) + (ad + bc)i) = (ac - bd)^2 + (ad + bc)^2 = (ac)^2 + (bd)^2 + (ad)^2 + (bc)^2 = (a^2 + b^2)(c^2 + d^2) = N(a + bi)N(c + di) \).

(II) is false since the units of \( \mathbb{Z}[i] \) are \( 1, -1, i, -i \).

(III) is false, however the converse is clearly true. As a counterexample, \( N(1 + 2i)|N(1 - 2i) \) but \( 1 + 2i \nmid 1 - 2i \).

(IV) is true. Suppose there exists non-units \( \alpha = a + bi \) and \( \beta = c + di \) such that \( p = \alpha \beta \). Then we have \( N(p) = p^2 = N(\alpha)N(\beta) \). Since neither \( \alpha \) nor \( \beta \) are units, we must have \( a^2 + b^2 = N(\alpha) = p^2 = 4k + 3 \). But this is clearly impossible since a sum of squares cannot be \( 3 \mod 4 \)

22. Since \( n^3 + 7n^2 - 13n + 19 = (n - 4)n^2 + (n - 4)11n + (n - 4)31 + 143 \) (synthetic division), then \( n - 4|n^3 + 7n^2 - 13n + 19 \to n - 4|413 \). Then \( n = 4 \to 1, 11, 13, 143 \to n = 3, 5, 15, 17, 147 \). The sum of all possible values of \( n \) is \( 3 + 5 + 15 + 17 + 147 = 187 \]

23. Using Euler’s Totient Theorem, we know that \( 3^{72} \equiv 1 \mod 91 \). Then \( 3^{391} \equiv (3^{72})^{5}3^{31} \equiv 3^{31} \mod 91 \). Then notice that,

\[
\begin{align*}
3 &\equiv 3 \mod 91 \\
3^2 &\equiv 9 \mod 91 \\
3^4 &\equiv -10 \mod 91 \\
3^8 &\equiv 9 \mod 91 \\
3^{16} &\equiv -10 \mod 91 \\
3^{32} &\equiv 9 \mod 91
\end{align*}
\]

\( 3^{32} \equiv 9 \mod 91 \) implies that \( 3^{31} \equiv 3 \mod 91 \)
24. From Vieta’s formula we know that \( p_1 + p_2 + p_3 = 40 \). If all the primes were odd then \( p_1 + p_2 + p_3 \) would be odd. Thus one of the primes must be 2. WLOG, let \( p_1 = 2 \). Then \( p_2 + p_3 = 38 \). Counting all the possible unordered triplets of \( (p_1, p_2, p_3) \) gives \((2, 7, 31)\) and \((2, 19, 19)\). Then the sum of all possible values for \(|c_2|\) is \( 2 \cdot 7 \cdot 31 + 2 \cdot 19 \cdot 19 = 1156 \). [E]

25. \[ \lim_{n \to \infty} \left( \frac{a_{n+1}}{a_n} = \frac{a_n}{a_{n-1}} \right) \to \]

\[ \frac{3a_n - a_{n-1}}{a_n} = \frac{a_n}{a_{n-1}} \]

\[ a_{n-1}(3a_n - a_{n-1}) = a_n^2 \]

\[ a_n^2 - 3a_n a_{n-1} + a_{n-1}^2 = 0 \]

\[ \left( \frac{a_n}{a_{n-1}} \right)^2 - 3 \frac{a_n}{a_{n-1}} + 1 = 0 \]

\[ a_n = \frac{3 \pm \sqrt{5}}{2} \]

By inspection the ratio is clearly greater than 1 so the ratio must be \( \frac{3 + \sqrt{5}}{2} \). [C]

26. We can examine the equation by various mods to determine possible values for \( x \). Taking mod 2 we see that \( 1^5 + 0^5 + 0^5 + x^5 \equiv 0^5 \mod 2 \to x \equiv 1 \mod 2 \). Taking mod 3 we see that \( 0^5 + 0^5 + 2^5 + x^5 \equiv 0^5 \mod 3 \to x \equiv 1 \mod 3 \). Taking mod 5 we see that \( 2^5 + (-1)^5 + 0^5 + x^5 \equiv (-1)^2 \mod 5 \to x \equiv 3 \mod 5 \). Taking mod 7 we see that \( (-1)^5 + 0^5 + (-2)^5 + x^5 \equiv 4^5 \mod 7 \to x \equiv 0 \mod 7 \). These conditions are enough to narrow the answer down to \( x \equiv 133 \mod 210 \). We can computationally verify that 133 satisfies this equation [A]

27. Using Sophie-Germaine factorization,

\[ 4^5 + 5^4 = 5^4 + 4 \cdot 4^4 \]

\[ = (5^2 + 2 \cdot 5 \cdot 4 + 2 \cdot 4^2)(5^2 - 2 \cdot 5 \cdot 4 + 2 \cdot 4^2) \]

\[ = (25 + 40 + 32)(25 - 40 + 32) \]

\[ = (97)(17) \]

Thus there are 2 prime factors of \( 5^4 + 4^5 \). [B]
28. First note that $3 \nmid x$. Next we add 1 to both sides and factor, which gives

$$(x + 1)(x^2 - x + 1) = 3^y$$

Let $x + 1 = 3^a$ and $x^2 - x + 1 = 3^b$ where $a + b = y$ and $b > a$. Then clearly $3^a \mid gcd(x-1, x^2+x+1)$ which is equivalent to $3^a \mid gcd(x-1, 3x)$ since $(x+1)^2 - (x^2-x+1) = 3x$. Then clearly we must also have $3^a \mid 3x$, but since $3 \nmid x$, we can only have $a = 0, 1$. These give the only solutions $(0, 0), (2, 2)$. Thus the sum is $0 + 0 + 2 + 2 = 4$ [A]

29. Since $ab = k^2$ we can say $a = n_1^2n_2$ and $b = n_2n_3^2$. But $a-b = n_1^2n_2 - n_2n_3^2 = n_2(n_1^2 - n_3^2) = p$. Thus $n_2$ must be 1. Then $n_1^2 = n_3^2 = (n_1 - n_3)(n_1 + n_3) = p$. So $n_1 - n_3 = 1$ and $n_1 + n_3 = p$. Then our possible pairs $(a, b)$ are $(4, 1), (9, 4), (16, 9), (36, 25), (49, 36)$. The sum of all such $a$ is $4 + 9 + 16 + 36 + 49 = 114$ [C]

30. Expanding gives

$$n^2 - 4n + 4 - m^2 + 4m - 4 = 2mn$$
$$n^2 - 4n - 2mn - m^2 + 4m = 0$$
$$n^2 + n(-4 - 2m) - m^2 + 4m = 0$$

Considering this as a quadratic equation in $n$ gives

$$n = \frac{4 + 2m \pm \sqrt{(-4 - 2m)^2 - 4(-m^2 + 4m)}}{2}$$
$$= \frac{4 + 2m \pm \sqrt{4m^2 + 16m + 16 + 4m^2 - 16m}}{2}$$
$$= \frac{4 + 2m \pm 2\sqrt{2m^2 + 4}}{2}$$

Now since $n$ must be an integer, this means that $\sqrt{2m^2 + 4}$ must be an integer. Then,

$$\sqrt{2m^2 + 4} = k \rightarrow k^2 - 2m^2 = 4$$

Notice that $k$ must be even so using the substitution $k = 2k_1$ gives $2k_1^2 - m^2 = 2$. Then by the same logic $m$ must also be even so using the substitution $m = 2m_1$ gives $k_1^2 - 2m_1^2 = 1$. This is a Pell’s equation with base solution $(3, 2)$, which gives the solution $(m, n) = (4, 12)$. The next smallest solution $(k_1, m_1)$ is given by

$$3^2 - 2 \cdot 2^2 = 1$$
$$(3 - 2\sqrt{2})(3 + 2\sqrt{2}) = 1$$
$$(3 - 2\sqrt{2})^2(3 + 2\sqrt{2})^2 = 1^2$$
$$(17 - 12\sqrt{2})(17 + 12\sqrt{2}) = 1$$

Thus $(k_1, m_1) = (17, 12) \rightarrow (m, n) = (24, 60)$. Then, $m + n = 24 + 60 = 84$ [D]