ANSWERS

1. 20
2. 16
3. 5
4. 4038
5. 462
6. 3198
7. 4037
8. $\frac{297}{625}$
9. $\frac{19}{3}$
10. 85
11. 4π
12. 6
13. 2,4,6,0
14. 2019
15. 6
16. 385
17. $\frac{2\sqrt{3}}{3\pi}$
18. -1
19. 193
20. $\frac{5}{12}$
21. 72
22. 6
23. $2\sqrt{2019}$
24. $\frac{1-5i}{26}$
25. $4\pi^2$

1. To find last three digits, do $\text{mod}1000 = -5^3\text{mod}1000 = -125\text{mod}1000 = 875$. Sum is 20.

2. Maximum area is equal legs of length $4\sqrt{2}$. Area is 16.

3. Clearly, 3 and 4 sides don't work. Let us imagine a pentagon sitting on a side. We can squish this from the top and stretch as necessary to create 3 angles as close to 180 degrees as we want.

4. First function is odd. Therefore, this is just $2019 + 2019 = 4038$.

5. We can perform stars-and-bars on 6 with 5 dividers such that it creates 6 regions where each region is a digit corresponding to how many stars are in it. There is $\binom{11}{5}$ ways to do this for a total of 462.

6. 3+673+1+2019=2696 403+80+16+3=502 2696+502=3198

7. Every person except the winner must have exactly two-losses by the end of the tournament. Then, the winner may have up to one loss and so the most is $2018 * 2 + 1 = 4037$.

8. The Doctor needs 3 intervals to go the 30 miles and Car-o-Line takes exactly 40 minutes to travel the 30 miles. Therefore, the Doctor must sleep at most one of four intervals. Our probability then is $\left(\frac{3}{5}\right)^3 + \left(\frac{2}{5}\right)\left(\frac{3}{5}\right)^3 + \left(\frac{3}{5}\right)\left(\frac{2}{5}\right)\left(\frac{3}{5}\right)^2 + \left(\frac{3}{5}\right)^2\left(\frac{2}{5}\right)\left(\frac{3}{5}\right) = \left(\frac{3}{5}\right)^3 + 3 * \left(\frac{2}{5}\right)\left(\frac{3}{5}\right)^3 = \frac{27}{125} * \left(\frac{11}{5}\right) = \frac{297}{625}$.

9. We look at each sharp point at $x = \frac{1}{3}$ and $x = \frac{7}{2}$. From $x = \frac{1}{3}$, if we step toward $\frac{7}{2}$ with step size k, we add on $3k$ from the first inequality and subtract $2k$ from the second, therefore we can only increase the value from the point where $x = \frac{1}{3}$ (similarly, we can see we decrease going toward 1/3 from 7/2). The value at this point is $7 - \frac{2}{3} = \frac{19}{3}$.

10. If the units digit is 2, then it is spaced exactly 30 apart every time for a total of 30 numbers. If the tens digit is 2, then for hundreds digit 1, 4, 7 there is 4 numbers and for other ones there is 3 numbers that are divisible by 3 for a total of 30 numbers, and if hundreds digit is 2 then we have 201 to 297 for a total of 33 numbers. Now we count for overlap, if any two digits are 2 and 2 then 2, 5, 8 make the number divisible by 3 so we subtract 9 then add back in 1 for 222 which these all overlap on for a total of $93 - 9 + 1 = 85$ numbers.

11. Area of a regular hexagon is $\frac{3s^2\sqrt{3}}{2} = 6\sqrt{3}$ so $s = 2$, which is the radius of the circle. Area is 4π.

12. Amy can always make the sum of Anthony and Amy's turn takeaways equal to 11. To make sure she wins, she takes until it is a multiple of 11, which is at 2013, so she takes 6 coins.

13. Either $|x - 3| = 1$ or $|x - 3| = 3$. Solving for the first we have $x = 4, 2$ and the second $x = 6, 0$.

14. If it converges, then it is true that $\lim_{n\to\infty} s_n = \lim_{n\to\infty} s_{n+1}$, so $\lim_{n\to\infty} s_n = \frac{s_n}{2019} + 2018$ so $\lim_{n\to\infty} s_n = 2019$.

15. Adding all three equations, we have $2a + 2b + 2c = 12$, so $a + b + c = 6$.

16. Use the formula $\frac{n(n+1)(2n+1)}{6} = 10 * 11 * \frac{21}{6} = 385$.

17. The diagonal of the cube is a diameter of the sphere's great circle. Volume of the cube is s^3 and sphere is $\frac{4}{3}\pi r^3$ and $s\sqrt{3} = 2r$. The ratio is then $\frac{\frac{8}{3\sqrt{3}}r^3}{\frac{4}{3}\pi r^3} = \frac{2}{\pi\sqrt{3}} = \frac{2\sqrt{3}}{3\pi}$.

18. $\frac{1}{k-2} - \frac{1}{k-3} = \left(\frac{1}{2} - 1\right) + \left(\frac{1}{3} - \frac{1}{2}\right) + \ldots = -1$

19. Notice that the remainders are half the divisor rounded up every time. We then try $\left\lceil \frac{5*7*11}{2} \right\rceil = 193$, which satisfies the constraints.

20. 7 is the exact midway for the outcome, with equal probability greater than 7 and less than 7. Since the probability for 7 is $\frac{1}{6}$ then our answer is $\frac{1-\frac{1}{6}}{2} = \frac{5}{12}$.

21. Clearly, we cannot have all three digits equal as then it would be sloped by given definition. For 2 equal digits, they must be both the first and last digit, and thus there is 5 numbers for repeated digits 2-7, 4 numbers for repeated 1 and 8 and 3 numbers for repeated 9, which gives us 41 numbers that satisfy this with repeated digits. For all distinct digits, we have 4 permutations of every 3 consecutive digits satisfying this (except for permutations of 210, which has 3). We have 8 of these, so we have $4 * 8 - 1 = 31$ from all distinct digits for a total of 72 total numbers.

22. $2^{2019} - 4 * 2^{2018} + 5 * 2^{2017} + 6 - 2^{2017} = (4 - 8 + 5 - 1) * 2^{2017} + 6 = 6$

23. Area increases by 4π every time starting at 4π. Our area for the 2019^{th} circle is then $4 * 2019\pi$, which has radius $2\sqrt{2019}$.

24. The denominator simplifies to $6 + 4i$, so we multiply both numerator and denominator by $3 - 2i$ to obtain $\frac{(1-i)(3-2i)}{2(3+2i)(3-2i)} = \frac{1-5i}{26}$.

25. One becomes the circumference of the base and the other the height. We can either then have $r = 1, h = 4\pi$ or $r = 2, h = 2\pi$, which results in volumes of $4\pi^2$ and $8\pi^2$ for a positive difference of $4\pi^2$.