Theta Bowl Solutions
MAΘ National Convention 2019

0. Since $216 = 6^3$, $a!$ must be divisible by both 2^3 and 3^3. The smallest such a is 9. Since $1000 = 5^3$, $b!$ must be divisible by both 2^3 and 5^3. The smallest such b is 15. The sum is $\mathbf{24}$.

1. A- The number must be a prime raised to 2nd power times one raised to the 4th power
$2^4 \bullet 3^2 = 144$
$3^4 \bullet 2^2 = 324$
$2^4 \bullet 5^2 = 400$
400 is our answer

B-
$2^6 \to 4$
$4^3 \to 2$
$8^2 \to 2$
$64^1 \to 1$
A total of 9

C-
$\sqrt[4]{11+2\sqrt{30}} = \sqrt{\sqrt{6}+\sqrt{5}}$
$\sqrt{6}+\sqrt{5}-\sqrt{5} = \sqrt{6} \to \sqrt[4]{6}$
answer is 6

400+9+6= 415

2. $0 = 3a^4 - 2a^2 - 1 = (3a^2 + 1)(a^2 - 1)$, so $a^2 = 1$, so $a = -1$.
$0 = 2b^{1/2} - 3b^{1/4} + 1 = (2b^{1/4} - 1)(b^{1/4} + 1)$, so $b^{1/4} = \frac{1}{2}$, so $b = \frac{1}{16}$.
Since $c, d \in \{0,1,2\}$, both quadratics have roots, so $c, d \neq 0$. If $c = 1$, then $d = 2$, and vice versa. Hence the product is $-\mathbf{\frac{1}{8}}$ or $\mathbf{-0.125}$.

3. $x^4 + 4 = (x^4 + 4x^2 + 4) - (4x^2) = (x^2 + 2)^2 - (2x)^2$
$= (x^2 + 2x + 2)(x^2 - 2x + 2)$, and
$x^4 + x^3 + x - 1 = (x^4 - 1) + (x^3 + x) = (x^2 - 1)(x^2 + 1) + x(x^2 + 1)$
$= (x^2 + x - 1)(x^2 + 1)$, so $a + b + c + d + e + f + g + h = \mathbf{5}$.

4. Let the appointment be in x hours. Suppose Alice would be y hours early going at 30 mph, and y hours late going at 20 mph. (The value of y is irrelevant.) Then she must travel $30(x - y) = 20(x + y)$ miles, so to arrive exactly on time, she must go at $\frac{2\cdot30(x-y)+3\cdot20(x+y)}{5x} = \frac{120x}{5x} = 24$ mph.
(Trivia: this is the harmonic mean of 30 mph and 20 mph.)

The initial solution has 1 mL of non-acid. Since this is 2% of the leftover solution, Bob must have extracted 50 mL of acid. The sum is $\mathbf{74}$.

5. The rational root theorem helps us factor: $(5x+4)(3x^2+x-3)=0$ and $(3y-4)(5y^2+3y+3)=0$. The only rational solutions are $x=-\frac{4}{5}$ and $y=\frac{4}{3}$, so $xy=-\mathbf{\frac{16}{15}}$.

6. If $x^2+5x+5=1$, then $(x+4)(x+1)=0$, so $x=-4$ or $x=-1$. If $x^2+5x+5=-1$, then $(x+3)(x+2)=0$, so $x=-3$. ($x=-2$ doesn't work.) Finally, if $x+5=0$, then $x=-5$. So $A=-13$.

   $0=\log(4-y^2)+\log(y^2)=\log(4y^2-y^4)$, so $4y^2-y^4=1$, so $y^4-4y^2+1=0$, so $y^2=\frac{4\pm2\sqrt{3}}{2}$. Since $0<y<2$ (or else the logarithms would be undefined), $y=\sqrt{\frac{4\pm2\sqrt{3}}{2}}=\frac{\sqrt{4\pm2\sqrt{3}}}{\sqrt{2}}=\frac{\sqrt{3}\pm1}{\sqrt{2}}$, so $B=\frac{2\sqrt{3}}{\sqrt{2}}=\sqrt{6}$. Finally, $A^2+B^2=\mathbf{175}$.

7. C_1 has diameter 5 (the hypotenuse of the right triangle). Because opposite angles of a cyclic quadrilateral are supplementary, the parallelogram must be a rectangle, so C_2 has diameter $\sqrt{5}$ (the diagonal of the rectangle). Similarly, the kite must have a right angle between a side of length 1 and a side of length 2, so C_3 also has diameter $\sqrt{5}$. The distance from the center of the hexagon to any vertex is 1 (since the hexagon can be dissected into six equilateral triangles), so C_4 has diameter 2. Finally, $abcd=\mathbf{50}$.

8. $a-3=\sqrt{a+\sqrt{a+\sqrt{a+\cdots}}}=\sqrt{a+(a-3)}$, so $(a-3)^2=2a-3$, so $0=a^2-8a+12=(a-6)(a-2)$, so (since $a>3$) $a=6$.

   $3-b=\sqrt{b+\sqrt{b+\sqrt{b+\cdots}}}=\sqrt{b+(3-b)}$, so $b=3-\sqrt{3}$.

   Hence $a+b=\mathbf{9-\sqrt{3}}$ or $\mathbf{-\sqrt{3}+9}$.

9. Setting $x = 0$ gives us $2f(0) + f(1) = 0$. Setting $x = 1$ gives us $2f(1) + f(0) = 1$. Subtracting the second equation from twice the first gives us $3f(0) = -1$, so $f(0) = -\frac{1}{3}$.

Let $a = g(0)$. Then $g(a) = g(g(0)) = 0$, so $g(g(a)) = g(0) = a$. Since $g(g(a)) = a^4 + 2a^3 - 2a^2 - 3a$, this gives us $0 = a^4 + 2a^3 - 2a^2 - 4a = a^3(a+2) - 2a(a+2) = a(a^2-2)(a+2)$. Since $(fg)(0)$ is rational and nonzero, we get $a = -2$, so $(fg)(0) = \mathbf{\frac{2}{3}}$.

(Note that such functions exist: $f(x) = x - \frac{1}{3}$ and $g(x) = x^2 + x - 2$ satisfy the conditions.)

10. There are nine points of intersection. From left to right, let $A, B,$ and C be the top three points, let $D, E,$ and F be the middle three points, and let $G, H,$ and I be the bottom three points. Then $m\angle GDE + m\angle EFI = 180$, so $\overleftrightarrow{AG} \parallel \overleftrightarrow{CI}$, so $(z) + (4z) = 180$, so $z = 36$. Also, $m\angle DEB = m\angle GHE$, so $\overleftrightarrow{DF} \parallel \overleftrightarrow{GI}$, so $(x + 90) + (z) = 180$, so $x = 54$. Finally, in quadrilateral $ABHG$, $(x + 45) + m\angle ABH + (3z) + (z) = 360$, so $m\angle ABH = 117$, so $y = 63$. The sum is $54 + 63 + 36 = \mathbf{153}$.

11. <u>First part:</u> Let $l = 2 + \sqrt{3}, w = 2 - \sqrt{3},$ and $h = 1$. Then $V = lwh = 1, S = 2(lw + lh + wh) = 2(1 + 2 + \sqrt{3} + 2 - \sqrt{3}) = 10$, and $d^2 = l^2 + w^2 + h^2 = 7 + 7 + 1 = 15$, so $A = \frac{(1)(10)}{15} = \frac{2}{3}$.

<u>Second part, hard method:</u> $(x^2 + y^2 + z^2) + 2(yz + zx + xy) = 25$, so (since $x, y, z > 0$) $x + y + z = \sqrt{25} = 5$. Since the sum of $x, y,$ and z is 5, the sum of their pairwise products is 5, and their product is 1, they must be the roots of $r^3 - 5r^2 + 5r - 1 = (r-1)(r^2 - 4r + 1)$. The smallest and largest roots are $2 - \sqrt{3}$ and $2 + \sqrt{3}$. Their sum is 4, so $AB = \mathbf{\frac{8}{3}}$.

<u>Second part, easy method:</u> Let $x, y,$ and z represent the dimensions of a rectangular prism. Then the volume is 1, the surface area is 10, and the square of the diagonal's length is 15. This is precisely the situation described

in the first part of the problem, so $x, y,$ and z are $2+\sqrt{3}$, $2-\sqrt{3}$, and 1, in some order. The sum of the smallest and the largest is 4, so $AB = \mathbf{\frac{8}{3}}$.

12. Since the entire triangle's angle measures must add to 180, $z = 70$. ΔABC and ΔCDE are isosceles, so $m\angle ACB = 50, m\angle ECD = 70$, and $m\angle CDE = 40$. Since $y = 180 - 50 - 70 = 60$ and ΔBCD is isosceles, $m\angle CBD = m\angle CDB = 60$. So finally, $x = 180 - 60 - 40 = 80$, and $x + y + z = \mathbf{210}$.

13. Let $O, E, F, G,$ and H be the points (0, 0), (0, 1), (2, 0), (0, -3), and (-4, 0), respectively. Then $\Delta OEF, \Delta OFG, \Delta OGH,$ and ΔOHE are right triangles, so $A, B, C,$ and D are the midpoints of the hypotenuses, so $ABCD$ is a parallelogram whose area is half the area of $EFGH$. The area of $EFGH$ is $\frac{1}{2}(6 \cdot 1 + 6 \cdot 3) = 12$, so the area of $ABCD$ is **6**.

14. Let $a = 2^x$ and $b = x^2$. Then $ab^2 + a^2 + b = ba^2 + b^2 + a$, so $0 = ba^2 - ab^2 + b^2 - a^2 + a - b = ab(a-b) - (a+b)(a-b) + a - b$ $= (ab - a - b + 1)(a - b) = (a-1)(b-1)(a-b)$, so $a = 1$, $b = 1$, or $a = b$. If $2^x = 1$, then $x = 0$. If $x^2 = 1$, then $x = \pm 1$. By graphing, we see that $2^x = x^2$ has three solutions ($x = 2, x = 4$, and one negative solution), so there are **6** solutions in total.

Since $(x^2 - x - 2) + (4 - x^2) + (x^2 + 3x + 2) = x^2 + 2x + 4$, we have $x^2 - x - 2 \geq 0$, $4 - x^2 \geq 0$, and $x^2 + 3x + 2 \geq 0$. The solution sets to these inequalities are $(-\infty, -1] \cup [2, \infty)$, $[-2,2]$, and $(-\infty, -2] \cup [-1, \infty)$. Their intersection is $\{-2, -1, 2\}$, so $A = 4$.

6+4=10