It is Wise, Generalize!

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Often it is the case that the solution of one problem enables us to propose several new problems. This may take the form of a converse problem, an extension of the original problem in one of several directions or an analogy. By all means do consider them. It provides us with more problems to solve on the one hand. On the other, we generally employ different solution strategies to solve the newly proposed problems so we gain greater perspectives and valuable insights. Let us look at a solution of the following problem in the light of preceding suggestions.

> (*) Find a sequence of distinct natural numbers with the property that the difference of squares of any two consecutive terms is a perfect cube.

Editor's Note: Try the problem yourself before reading on!

Solution. If we assume $s_1, s_2, \ldots$ as the sequence of natural numbers to be determined, then we must have

$$s_{r+1}^2 - s_r^2 = t_r^3, \qquad r = 1, 2, 3, \ldots.. \tag{1}$$

where t's are also natural numbers. When $r = 1$ then $s_2^2 - s_1^2 = t_1^3$. A little experimentation shows that $3^2 - 1^2 = 2^3$. So we take $s_1 = 1$, $s_2 = 3$, $t_1 = 2$. When $r = 2$ then $s_3^2 - s_2^2 = t_2^3$ holds. With $s_2 = 3$, we find that $s_3 = 6$. Likewise we find that $s_4 = 10$. Let us write down the initial terms of the sequence:

$$1, 3, 6, 10, s_5, s_6, \ldots$$

It is now easy to guess that $s_5 = 15$, $s_6 = 21$, and that $s_r = r(r + 1)/2$. By using the induction principle we can convince ourselves that

$$s_r = r(r + 1)/2 \qquad r = 1, 2, 3, \ldots \tag{2}$$

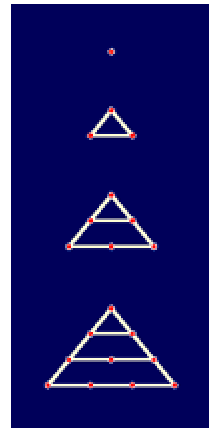
is the correct formula for a solution to our problem. It is hoped that the reader will complete the induction step.

A Pythagorean Concept

Hindsight tells us that (*) was a relatively easy problem to solve. If we think so and forget all about it then much of the mathematical pleasure and the thrill of mathematical discovery will be lost. After solving the problem (*) one question we might ask ourselves is this: Does the sequence

$$1, 3, 6, 10, \ldots, r(r + 1)/2, \ldots \tag{3}$$

have any special significance? The formula (2) might give us a clue: $1 = 1$, $3 = 1+2$, $6 = 1 + 2 + 3$, $10 = 1 + 2 + 3 + 4$, . . . , $r(r + 1)/2 = 1 + 2 + 3 + . . . + r$, and so s_r is the sum of the first r natural numbers for $r = 1, 2, 3, . . .$ Ancient Pythagoreans, students of Pythagoras - the name attached to the famous theorem on the squares of the sides of a right-angled triangle - noticed the preceding property and more. They discovered that they could form a triangle figure for each term of (3) using that same number of objects. Here it goes: The triangle figure for



$s_1 = 1$ contains one object on each side. Here we say the triangle shrinks to a point.

$s_2 = 3$ contains two objects on each side. The total number of objects used is 3.

$s_3 = 6$ contains three objects on each side. The total number of objects used is 6. Notice that this triangle figure is an extension of the one for s_2.

$s_4 = 10$ contains four objects on each side. The total number of objects now used is 10. Notice again that the present triangle figure is an extension of the previous one for s_3.

The reader may depict a triangle figure using a total of 15 or 21 objects. The figure for s_r contains r objects on each side and a total of $r(r + 1)/2$ objects. So the ancient Pythagoreans natrually called

$$1, 3, 6, 10, . . ., r(r + 1)/2, . . .$$

the sequence of triangular numbers. For future reference we restate the property observed earlier:

Each triangular number is a sum of an arithmetic progression whose first term is 1 and the common difference is 1. (4)

Generalization to *n*-gonal Numbers

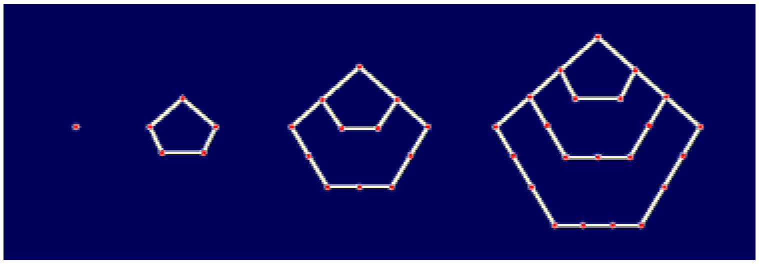
Curiosity compels us to ask and answer a host of questions such as

(i) What sequence of numbers might be called a sequence of square numbers? The answer is built into the question itself: 1, 4, 9, 16, ..., r^2, ... Try to depict a sequence of square figures using a total of 1, 4, 9, 16, ... objects respectively. Discover a property of square numbers analogous to (4) in the solution of the following problem:

> Find the arithmetic progression whose sequence of sums is the sequence of square numbers. (The sequence intended is the one formed by taking the sum of the first 1 term, then the first 2 terms, then the first 3 terms, etc.)

The answer is a well known property of positive odd integers. Our discussion has provided a new meaning to it while placing it in the natural context of development of the concept of *n*-gonal numbers.

(ii) What sequence of numbers might be called a sequence of pentagonal numbers? The following pentagonal figures should help deciphering the first few pentagonal numbers - simply count the total number of objects in each figure.



Deduce the formula $P(5,r) = r(3r - 1)/2$ for the rth pentagonal number. Find the arithmetic progression whose sequence of sums is the sequence of pentagonal numbers.

The success with the triangular, the square, and the pentagonal numbers should encourage us to find the sequence, the formula, and the arithmetic progression whose sums will yield the sequence in the case of hexagonal, heptagonal, octagonal,..., *n*-gonal numbers, which are also called *polygonal* numbers. This is one direction of thought that we pursued. Are there others?

Further Directions

Let us revisit problem (*) for further inspiration.

I. We obtained the sequence of triangular numbers as a solution to the problem (*). Can we find a different, finite or infinite sequence of distinct natural numbers that is also a solution to the problem (*)? Note that a subset of (3), finite or infinite, such as {15,21,28,36} or {15,21,28,36,45,...} cannot be considered a distinct solution to the problem (*).

II. We might also propose an analogous problem. Find a sequence of distinct natural numbers such that the difference of squares of any two consecutive terms is (a) a perfect square, (b) a perfect biquadrate (fourth power).

Finally, try to pose and solve other variants of the problem (*). Of course, we should be aware that posing such a problem does not guarantee a solution to it. It is the mathematical experience that matters most!

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