1. A Simply plugging in $s = 4$ yields $a(4) = \frac{20}{23+\sqrt{4}} = \frac{20}{25} = \frac{4}{5} = \mathbf{80}\%$

2. A Notice that pairs of trigonometric functions cancel (i.e., $\sin(t)\csc(t) = 1$, etc.) such that $f(t) = 1 - 1 = 0$. Therefore, $f\left(\frac{\pi}{2023}\right) = \mathbf{0}$.

3. C We have $\cos(t) = \frac{x}{5}$ and $\sin(t) = \frac{y}{3}$. Using the Pythagorean identity to eliminate the parameter t, $\cos^2(t) + \sin^2(t) = \frac{x^2}{25} + \frac{y^2}{9} = 1$. This is the equation of an ellipse with semi-major axis length 5 and semi-minor axis length 3, and with area $5 \cdot 3 \cdot \pi = \mathbf{15\pi}$.

4. D Let the distance be c. Using law of cosines, we have $c^2 = 8^2 + 5^2 - 2(8)(5)\cos\left(\frac{\pi}{3}\right) = 49$, and therefore $c = \mathbf{7}$.

5. B As $|\vec{u} \cdot \vec{v}| = |\vec{u}||\vec{v}|\cos(\theta)$ and $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}|\sin(\theta)$, and $\cos\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right)$, we have $|\vec{u} \cdot \vec{v}| - |\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}|\left(\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\right) = \mathbf{0}$.

6. A Their cumulative earnings after the 5th day are $5^3 + 5 + 1 = 131$. Their cumulative earnings after the 4th day are $4^3 + 4 + 1 = 69$. Therefore, they made $131 - 69 = \mathbf{62}$ during their 5th day.

7. A A cardioid is of the form $r = a \pm a\sin(\theta)$ or $r = a \pm a\cos(\theta)$. Of the answer choices, $\boldsymbol{r = 3 + 3\sin(\theta)}$ satisfies this form.

8. C By nature of skipping stones, once a stone fails a skip, it sinks and can no longer skip anymore. Therefore, if Emma’s stone skips at least 3 times, it must have successfully completed its first three skips. By the multiplication principle, this happens with probability $\sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right) = 1 \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} = \mathbf{\frac{\sqrt{6}}{4}}$.

9. D A matrix is singular (un-invertible) if it has determinant zero. The determinant of the “space” matrix is $4x^2 - (2-y)(2+y)$. For this to be zero, we have $4x^2 - (4 - y^2) = 0 \rightarrow 4x^2 + y^2 = 4$. This is the equation of an ellipse with major axis of length 4 and minor axis of length 2, which encloses an area of $\mathbf{2\pi}$.

10. C Let $a(t) = S(t) - T(t) = \sqrt{t + \sqrt{2t-1}} - \sqrt{t - \sqrt{2t-1}}$. Then $[a(t)]^2 = t + \sqrt{2t-1} + t - \sqrt{2t-1} - 2\sqrt{(t+\sqrt{2t-1})(t-\sqrt{2t-1})} = 2t - 2\sqrt{t^2 - 2t + 1} = 2t - 2|t-1| = 2$. Therefore, $S(t) - T(t) = \sqrt{2}$. Evaluated at $t = \frac{23}{20}$, this is still equal to $\sqrt{2} \approx 1.414$, which is closest to **3/2**.

11. D In this context, the x-coordinate of a point on the unit circle is $\cos(\theta)$ and the y-coordinate of a point on the unit circle is $\sin(\theta)$, where θ is the positive angle the arrow makes with the x-axis. If Legosi spins the arrow through an angle of α with his left hand and an angle of β with his right, then the cumulative angle would be $\alpha + \beta$. We wish to find $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) = \left(\frac{5}{13}\right)\left(-\frac{3}{5}\right) - \left(\frac{12}{13}\right)\left(\frac{4}{5}\right) = -\mathbf{\frac{63}{65}}$.

12. C This is the common volume between a sphere with radius 1 and its inscribed octahedron: $\frac{4\pi}{3}(1)^3 - 2(2)(1)\left(\frac{1}{3}\right) = \boldsymbol{\frac{4}{3}(\pi - 1)}$.

13. C The number of positive divisors of a number with prime factorization $p_1^{e_1} \cdot p_2^{e_2} \cdots p_n^{e_n}$ where p_i is prime, is $(e_1 + 1)(e_2 + 1) \cdots (e_n + 1)$. For there to be 6 positive factors, the product must be either $(5 + 1)$ or $(2 + 1)(1 + 1)$. For the $(5 + 1)$ case, the smallest possible number is $2^5 = 32$. For the $(2 + 1)(1 + 1)$ case, the smallest possible number is $2^2 \cdot 3^1 = 12$. Therefore, this first occurs on November 12th. As November 1st is a Sunday and there are 7 days in a week, the 12th is on a **Thursday**.

14. B This is a downward-facing parabola, whose vertex occurs at $t = -\frac{b}{2a} = \frac{-240}{-32} = \frac{15}{2}$. Plugging in, this yields a height of $-16\left(\frac{15}{2}\right)^2 + 240\left(\frac{15}{2}\right) + 10 = \mathbf{910}$.

15. D The midpoint of the object's oscillation is $C = \frac{100+40}{2} = 70$. The amplitude is $A = 100 - 70 = 30$. The period of the object is $6 \rightarrow \frac{2\pi}{B} = 6 \rightarrow B = \frac{\pi}{3}$. The object starts at height 100 and immediately goes down. Therefore, our height function is $h(t) = 30\cos\left(\frac{\pi t}{3}\right) + 70$, and $h(11) = 30\cos\left(\frac{11\pi}{3}\right) + 70 =$ **85**.

16. C Expressing this with the first few banks, our sum is $1000 + 900 + 810 + \cdots$. This can be evaluated using the infinite geometric series formula with a common ratio of 0.9. Therefore, the sum is $\frac{1000}{1-0.9} = \mathbf{10,000}$.

17. B Call T the temperature in Kelvin and P the pressure. As pressure varies directly with absolute temperature (in Kelvin), $\frac{P_1}{T_1} = \frac{P_2}{T_2}$. Plugging in variables, $\frac{3.5 \text{ atm}}{12+273} = \frac{P_2}{99+273}$. This yields $P_2 = 372 \cdot \frac{3.5 \text{ atm}}{285} \cdot \frac{760 \text{ mmHg}}{1 \text{ atm}} \cdot \frac{1 \text{ torr}}{1 \text{ mmHg}} = \mathbf{3472}$.

18. C From the initial value of 16 to a final value of 2, the amount has been halved 3 times. This happened over 9 days, so the half-life is $t_{1/2} = \frac{9}{3} = \mathbf{3}$.

19. C Recall that if you multiply a row or column of a matrix by a constant, the determinant is also multiplied by that constant, if you swap a row/column with another row/column, then the determinant is negated, if a constant multiple of a row/column is added to another row/column, the determinant is unaffected, and the determinant of the inverse is the inverse of its determinant. Therefore, our answer is $\left(3 \cdot \left(\frac{1}{2}\right)^3 \cdot (-1)^2 \cdot \left(\frac{1}{3}\right) \cdot (-1)^3\right)^{-1} = \mathbf{-8}$.

20. A As $P(x)$ has real coefficients, we may use the conjugate root theorem for complex numbers: as $1 + 2i$ is a root, $1 - 2i$ is also a root. However, the rational root theorem does not apply as this would require that all coefficients must be rational (the question states real). As the sum of the roots is given to be $4 - \sqrt{2}$, the final root must be 1. This means that $P(1) = 0$. The sum of the coefficients of a polynomial is $P(1)$, so our answer is **0**.

21. A Two chords determine 4 distinct points on a circle. We rephrase the problem into: "given any 4 distinct points, what is the probability that the chords formed by pairs of these points intersect?" Through simple experimentation of all possible pairings (three total), it is clear this probability is $\mathbf{\frac{1}{3}}$.

22. B For two chords to be drawn in the framework of this question, they must be formed by four distinct points or three distinct points (sharing one endpoint). In the former case, the probability that the chords intersect inside the circle is 1/3 (from the previous problem). In the latter case, the chords will never intersect (probability 0). For two chords to share an endpoint: after the first chord is drawn, there are $\binom{6-2}{2}$ ways to choose the second chord, and $\binom{6}{2} - 1$ total ways to choose a distinct second chord. The probability that two chords share an endpoint is $\frac{\binom{6-2}{2}}{\binom{6}{2}-1} = \frac{4}{7}$. The total probability that two chords intersect is therefore $\frac{4}{7}(0) + \left(1 - \frac{4}{7}\right)\left(\frac{1}{3}\right) = \mathbf{\frac{1}{7}}$.
*Alternatively, there are $\binom{6}{4} = 15$ ways to pick 4 points on the circle. Since there is 1 way to choose line segments such that those intersect, the probability is

$$\frac{15}{\binom{\binom{6}{2}}{2}} = \frac{15}{105} = \mathbf{\frac{1}{7}}$$

23. A The robbers and police are 10 movements apart (5 horizontal, 5 vertical), and therefore must meet on their 5th movement if they are to run into each other. The potential meeting spots are thus (0,5), (1,4), (2,3), (3,2), (4,1), and (5,0). To reach (0,5) or (5,0), each group of robbers and police will have $\binom{5}{0} = 1$ path, $\binom{5}{1} = 5$ paths for (1,4) and (4,1), and $\binom{5}{2} = 10$ paths for (2,3) and (3,2). There are 2^5 total paths for each case. Therefore, the probability they will both end up at the same location is $2\left(\left(\frac{1}{32}\right)^2 + \left(\frac{5}{32}\right)^2 + \left(\frac{10}{32}\right)^2\right) = \mathbf{\frac{63}{256}}$.

24. A. In this ellipse, $a = \sqrt{49} = 7$ and $c = \sqrt{49 - 33} = 4$, so the foci are at $(-4,0)$ and $(4,0)$. Assume that Mitsuha starts from the $(-4,0)$ focus. From the geometric definition of an ellipse, the sum of the distances from a point on the ellipse to both foci are a constant $2a = 14$. Therefore, the length of Mitsuha's entire journey is always 14. We wish to minimize the time Mitsuha spends after coming in contact with the circle, or in other words to minimize the distance between Mitsuha's starting foci to the edge of the circle. This occurs when her path crosses through the center of the circle $(0,3)$. This minimum value is the distance from $(-4,0)$ to $(0,3)$ minus the radius of the circle, or $5 - 2 = 3$. Therefore, the maximum length Mitsuha can travel after coming into contact with the circle is $14 - 3 = \mathbf{11}$.

25. D Numbers which have a remainder of 2 when divided by 5 can be expressed in the form $5a + 2$, where a is an integer. Similarly, numbers that have a remainder of 5 when divided by 7 can be expressed as $7b + 5$. Looking at $7b + 5$, the 5 is already divisible by 5, which means that $7b$ must have a remainder of 2 when divided by 5 in order to satisfy the first condition. As $7 \equiv 2 \pmod{5}$, we need $b \equiv 1 \pmod{5}$. Let's call $b = 5c + 1$. Plugging this in, Megumin is looking for numbers of the form $7(5c + 1) + 5 = 35c + 12$. This is an arithmetic sequence with the first term being 12 at $c = 0$ and largest three-digit ("final") term being 992 at $c = 28$. There are $28 - 0 + 1 = 29$ terms, an odd number. Therefore, the median is $\frac{12+992}{2} = \mathbf{502}$.

26. C Once they reach deuce, let p be the probability of Alex winning the game. Alex can win two points consecutively (probability $\frac{1}{10}\cdot\frac{1}{10}=\frac{1}{100}$) to win the game. He can also win one and lose one with probability $2\cdot\frac{1}{10}\cdot\frac{9}{10}=\frac{9}{50}$ and return the game to deuce, after which he yet again has a probability p of winning the game. Therefore, we have the equation $p=\frac{1}{100}(1)+\frac{9}{50}(p)\rightarrow p=\mathbf{\frac{1}{82}}$.

27. A Let V_n be the number of n digit Vincent Numbers. For an n digit Vincent Number, there are two possible cases: it either ends in a 4 or a 5.
   - Case 1: If an n digit Vincent Number ends in a 4, then the number can be formed by a $(n-2)$ digit Vincent Number with the $(n-1)$th digit necessarily being a 5. There are V_{n-2} ways to do this.
   - Case 2: If an n digit Vincent Number ends with a 5, then there is no restriction on the $(n-1)$th digit as long as the number up to the $(n-1)$th digit is a Vincent Number. There are therefore V_{n-1} possible combinations.

Combining our two cases, we see can establish the recursive formula $V_n=V_{n-1}+V_{n-2}$. Through simple counting, it is easy to see that $V_1=2$ and $V_2=3$. Using the recursive formula from there (i.e., $V_3=2+3=5$, and so on), we have $V_{10}=\mathbf{144}$.

*Note that Case 1 and Case 2 are mutually exclusive

28. B From the previous question, $V_n=V_{n-1}+V_{n-2}$ with $V_1=2$ and $V_2=3$. Writing out the first terms, we have

$$S=\frac{2}{3}+\frac{3}{9}+\frac{5}{27}+\frac{8}{81}+\cdots$$

Similar to an arithmetic-geometric sequence, we multiply both sides of this equation by 3 and subtract from each other:

$$3S=2+\frac{3}{3}+\frac{5}{9}+\frac{8}{27}+\cdots$$

$$\rightarrow 3S-S=2S=2+\frac{1}{3}+\frac{2}{9}+\frac{3}{27}+\frac{5}{81}=2+\frac{1}{3}+\frac{S}{3}$$

This yields $2S=2+\frac{1}{3}+\frac{S}{3}$, or $S=\mathbf{\frac{7}{5}}$.

29. D Interpreting the equation geometrically, it describes the locus of all z such that the distance from z to 4 plus the distance from z to -4 is a constant value of 10. This describes an ellipse with foci at -4 and 4 ($c=4$) and semi-major axes $2a=10\rightarrow a=5$. The semi-minor axis is thus $\sqrt{25-16}=3$, and the eccentricity is $e_1=\frac{4}{5}$. To "isolate" z, we parametrize this ellipse in the Argand plane much like in the Cartesian plane by writing $z=5\cos(\theta)+3i\sin(\theta)$. Applying our transformation $z\rightarrow z^2$, the set of z^2 can be written as $25\cos^2(\theta)+30i\sin(\theta)\cos(\theta)-9\sin^2(\theta)=17\cos^2(2\theta)+15i\sin(2\theta)+8$. As we may write $2\theta=\alpha$ for the parameter, this simply describes another ellipse centered at 8 with $a=17$, $b=15$, and $c=\sqrt{17^2-15^2}=8$ and eccentricity $e_2=\frac{8}{17}$. Therefore, $\frac{e_2}{e_1}=\frac{8}{17}\cdot\frac{5}{4}=\frac{10}{17}\rightarrow 10+17=\mathbf{27}$.

30. A By simply playing around with a few circles, it is clear that the majority of points are "connected," and that between any two circles the boundary of "connected" points are

determined by exterior tangents. With many more circles, it is easier to use complimentary counting and find the total length of "unconnected" perimeter. Consider three circles and one lying in the interior formed by their triangle. By drawing all sets of external tangents to the central circle, it is clear there are no unconnected points on this central circle. Similarly, a circle lying directly in the middle of two others has no unconnected points. Therefore, in a group of 2023 circles, the only ones which may yield any unconnected perimeter lie on the "border" circles. Imagine wrapping a string tightly around the group of circles (such that it only goes around the border ones). An ant which walks one loop on this string would turn exactly $360°$, and only turns while walking along portions of the string hugging a circle (not the straight "tangent" sections). Therefore, there exists exactly one circle's worth of unconnected perimeter, or π. There is a total of $2\pi\left(\frac{1}{2}\right) \cdot 2023 = 2023\pi$ total perimeter, so the perimeter of "connected" points is $2023\pi - \pi = \boldsymbol{2022\pi}$.