Alpha Cipher Nationals 2024 solutions

0. $\left(\cos^2 x+\sin^2 x\right)\left(\cos^2 x-\sin^2 x\right)=\left(\cos^2 x-\sin^2 x\right)=\cos 2x \to \frac{2\pi}{2}=\pi$

1. $2^{14}+1=\left(2^7+1\right)^2-2^8=\left(2^7+1-2^4\right)\left(2^7+1-2^4\right)=113\bullet 145=113\bullet 5\bullet 29\to 113+29=142$

2. $\frac{L-40}{40}=\frac{L-56}{36}\to 36L-40(36)=40L-40(56)$

$4L=40(56-36)\to L=200$

3. Based on the position of the foci, the center is the origin so h and k are irrelevant.

a/b is the slopes of the asymptotes which is 2, so a =2b

$c^2=25=a^2+b^2\to a=2b\to a^2=4b^2\to 25=5b^2\to b^2=5$ therefore;

$a^2=20\to 20-5=15$

4. $\lim_{x\to\infty}(\sqrt{2x^2+x+7}-\sqrt{2x^2-x})\bullet\frac{(\sqrt{2x^2+x+7}+\sqrt{2x^2-x})}{(\sqrt{2x^2+x+7}+\sqrt{2x^2-x})}=\frac{2x^2+x+7-2x^2+x}{(\sqrt{2x^2+x+7}+\sqrt{2x^2-x})}$

$\frac{2x+7}{(\sqrt{2x^2+x+7}+\sqrt{2x^2-x})}\to\frac{2}{2\sqrt{2}}=\frac{\sqrt{2}}{2}$

5. Law of cosines: $36=4x^2+x^2-2\bullet 2x\bullet x\cos 120^\circ\to 36=5x^2+2x^2=7x^2$

$2x^2=\frac{72}{7}$

Note that the other solution, $36+x^2+12x=4x^2\to 3x^2-12x+36=0\to x=-1+\sqrt{13}$ does not work since the longest side is 6 as $6>2\cdot(-1+\sqrt{13})$.

6.
$$r^2 = 50 + 150 = 200 \to r = 10\sqrt{2}$$
$$\tan\theta = \frac{y}{x} = \sqrt{3} \to \theta = \frac{\pi}{3} \to \left(10\sqrt{2}, \frac{\pi}{3}\right) \to \left(10\sqrt{2}, \pi\right) \to \left(-10\sqrt{2}, 0\right) \to -10\sqrt{2}$$

7. $999 -_9 C_1 -_9 C_2 -_9 C_3 = 999 - 9 - 36 - 84 = 870$

8. classic ellipse question. 2a=20 so a=10. 2c=10 so c=5
$$b^2 = 100 - 25 = 75 \to b = 5\sqrt{3} \to A = \pi ab = 50\pi\sqrt{3}$$

9.
$$y = \frac{3\sin x\cos^2 x + \cos^4 x\sin x + 3\sin^3 x + \cos^2 x\sin^3 x}{\sec x\tan x - \sin x\tan^2 x}$$
$$y = \frac{3\sin x\left(\cos^2 x + \sin^2 x\right) + \sin x\cos^2 x\left(\cos^2 x + \sin^2 x\right)}{\tan x\left(\frac{1-\sin^2 x}{\cos x}\right)}$$
$$y = \frac{\sin x\left(3 + \cos^2 x\right)}{\sin x} = 3 + \cos^2 x \to (3,4) \to 12$$

Note that the endpoints are unachievable as $x \neq \frac{k\pi}{2}$

10. There are 6 increasing with common difference of 1(1234, 2345,…6789). There are 3 increasing with common difference of 2(1357,2468,3579). These 9 can be flipped to be decreasing as well for a total of 18. But we also have 3 decreasing ones that can end in 0(9630,6420,3210) for a grand total of 21.

11.
$$2\arctan\frac{1}{3} = \arctan k - \frac{\pi}{4} \to k = \tan\left(\frac{\pi}{4} + 2\arctan\frac{1}{3}\right)$$
$$k = \frac{1 + \tan\left(2\arctan\frac{1}{3}\right)}{1 - \tan\left(2\arctan\frac{1}{3}\right)} \to \tan\left(2\arctan\frac{1}{3}\right) = \frac{\frac{1}{3} + \frac{1}{3}}{1 - \frac{1}{3}\bullet\frac{1}{3}} = \frac{3}{4}$$
$$k = \frac{1 + \frac{3}{4}}{1 - \frac{3}{4}} = 7$$

12. Draw the picture and then connect the centers and you get an equilateral triangle of side 6. Draw a segment from a vertex to the center of the smaller circle and then draw an altitude down. This creates a 30-60-90 triangle with segments of $\sqrt{3}$,3,2 $\sqrt{3}$. Our radius is $2\sqrt{3}-3$

Answers:

0. π
1. 142
2. 200
3. 15
4. $\frac{\sqrt{2}}{2}$
5. $\frac{72}{7}=10\frac{2}{7}$
6. $-10\sqrt{2}$
7. 870
8. $50\pi\sqrt{3}$
9. 12
10. 21
11. 7
12. $2\sqrt{3}-3$