Unless otherwise noted, assume the following

- When a complex number is written in component form, such as $z = a + bi$, a, b are real.
- Further, $Re(z) = a$ and $Im(z) = b$.
- When a complex number is written in polar/exponential form, such as $z = r \operatorname{cis} \theta = re^{i\theta}$, r is real and non-negative, and θ is the principal argument, namely, $\theta \in (-\pi, \pi]$.
- $\sqrt{z}$ denotes the principal square root of z.
- $\ln z$ uses the principal argument of z.
- $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ the set of reals, and $\mathbb{C}$ the set of complex numbers.

1. Evaluate:

$$\sum_{n=-10}^{10} ni^n$$

A. $10 - 10i$ B. $10 + 10i$ C. $10i$ D. 10 E. NOTA

2. Evaluate $|(2+i)^{10} \cdot (3+4i)|$

A. 15625 B. 3125 C. 625 D. 25 E. NOTA

3. Find the positive difference between the arguments of $-1 - 4i$ and $-4 + i$.

A. $\frac{\pi}{2}$ B. $\frac{3\pi}{2}$ C. $\frac{\pi}{4}$ D. $\frac{5\pi}{4}$ E. NOTA

4. If $x + \frac{1}{x} = 2\cosh(\theta)$, find $x^2 + \frac{1}{x^2}$.

A. $2\sinh(2\theta)$ B. $\sinh(2\theta)$ C. $\cosh^2(\theta)$ D. $\sinh^2(\theta)$ E. NOTA

5. The roots to $x^5 = 3125$ are graphed on the Argand Diagram. When connected, they form pentagon $MRZLU$. Compute the product of the distances from M to every other point on the pentagon.

A. 5 B. $125\sqrt{3}$ C. 3125 D. $3125\sqrt{3}$ E. NOTA

6. Simplify $\frac{(5+12i)^{31} \cdot (1+i\sqrt{3})}{(-119+120i)^{15} \cdot (\sqrt{3}-i)}$

A. $5 + 12i$ B. $-12 + 5i$ C. $9 + 40i$ D. $-40 + 9i$ E. NOTA

7. Which of the following is closest to the measure, in radians of $\arg(100+i)$

A. 0 B. $\frac{1}{100}$ C. π D. $\frac{\pi}{100}$ E. 1

For questions 8-10, refer to the following:

Consider a modified version of the buzzing game Science Bowl: 2 teams, called Team A and Team B, attempt to correctly answer questions read by a moderator. There are 10 questions read that can be answered by either team, called toss-up questions. If a team gets one of these toss-ups correct, they receive 4 points, and an additional question called a bonus question. Only the team that correctly answered the toss-up can answer the bonus, and if the bonus is answered correctly, $10i$ points are added to the team score of that team. It is not possible to lose points. The goal of the game is to maximize the magnitude of the score, and the team which has the score of higher magnitude at the end of the 10 toss-up questions (and anywhere from 0 to 10 bonus questions) wins the game.

8. Team 1 gets 7 toss-up questions and 2 bonus questions out of the 7 bonus questions they receive. Team 2 gets 3 toss-up questions and all 3 bonus questions they receive. What is the magnitude of the score of the winning team?

A. $3\sqrt{97}$ B. $4\sqrt{74}$ C. $6\sqrt{29}$ D. $9\sqrt{22}$ E. NOTA

9. How many toss-up questions must Team A answer correctly in order to be certain of winning, regardless of any other factors?

A. 6 B. 7 C. 8 D. 9 E. NOTA

10. At the end of 10 questions, both teams are perfectly tied. How many possible scores are there? For example, if both teams got 1 toss-up and the associated bonus, they have a score of $4+10i$, so $4+10i$ is a possible final score of both teams.

A. 5 B. 10 C. 15 D. 21 E. NOTA

11. On the Argand Diagram, find the area the region enclosed by the graph of $|z|+|z+1|=2$.

A. 2π B. $\pi\sqrt{3}$ C. π D. $\frac{\pi\sqrt{3}}{2}$ E. NOTA

12. Let the 8 solutions to $z^8 - 1 = 0$ be $a_n = x_n + y_n i$ for $n = 1, 2, 3, \dots, 8$. They are graphed on the Argand Diagram along with a 9th point $a_9 = x_9 + y_9 i$. a_9 starts at $2 + 0i$ and is rotated around the origin in a full circle. Let b be the arithmetic mean of $a_1, a_2, \dots, a_9$. b traces out a path as a_9 rotates. What is the area of the figure enclosed by the path?

A. 4π B. $\frac{4\pi}{9}$ C. $\frac{2\pi}{9}$ D. $\frac{4\pi}{81}$ E. NOTA

13. $f: \mathbb{R} \to \mathbb{C}$ is a function that satisfies $2f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$.
If $f\left(\frac{\pi}{6}\right) = -\sqrt{3} - i$, compute $f\left(\frac{11\pi}{3}\right)$.

A. $1 + i\sqrt{3}$ B. $1 - i\sqrt{3}$ C. $-1 + i\sqrt{3}$ D. $-1 - i\sqrt{3}$ E. NOTA

14. Consider the polar graph $r = \theta$, where θ is in the domain of real numbers. A point on this graph is at an angle θ coterminal with $\frac{\pi}{3}$ but not equal to $\frac{\pi}{3}$. What is the smallest possible distance to the pole of this point?

A. $\frac{\pi}{3}$ B. $\frac{7\pi}{3}$ C. $\frac{13\pi}{3}$ D. $\frac{19\pi}{3}$ E. NOTA

15. Compute $\lim_{x \to i} \frac{\sqrt{x^2+1}}{x-i}$.

A. DNE B. 1 C. $\frac{\sqrt{2}+i\sqrt{2}}{2}$ D. i E. NOTA

16. Let x and y be complex numbers such that $x = a + bi$ and $y = c + di$ where $a, b, c, d \in \mathbb{R}$. Given that $3x + 4y = 5 + 10i$, and $Re(x + yi) = 1$, what is the value of $9b - 16c$?

A. -2 B. 6 C. 14 D. 22 E. NOTA

17. Compute $\arg(20 + 21i)$.

A. $\arcsin\left(\frac{21}{29}\right)$ B. $\arccos\left(\frac{20}{21}\right)$ C. $\arctan\left(\frac{20}{21}\right)$ D. $\operatorname{arccsc}\left(\frac{21}{29}\right)$ E. NOTA

18. Let $z^2 = 4 + 2i\sqrt{5}$. If $z = a + bi$, where $a, b \in \mathbb{R}$ and $a > 0$, compute $a + b\sqrt{5}$.

A. 2 B. 4 C. $2\sqrt{5}$ D. $4\sqrt{5}$ E. NOTA

19. Which of the following is the value of $\ln(1 - i\sqrt{3})$? (Check the directions at the top of the test if needed)

A. $2 + \frac{\pi i}{3}$ B. $\ln(2) + \frac{\pi i}{3}$ C. $2 - \frac{\pi i}{3}$ D. $\ln(2) - \frac{\pi i}{3}$ E. NOTA

20. Evaluate

$$\prod_{n=-90}^{90} cis(n^\circ)$$

A. -1 B. $\frac{(\sqrt{2}+i\sqrt{2})}{2}$ C. $\frac{(\sqrt{2}-i\sqrt{2})}{2}$ D. i E. NOTA

For questions 21-23, $f(x) = 2^x - x^2$

21. Two of the roots of $f(x)$ lie on the positive real axis of the Argand Diagram. What is the distance between these roots?

A. 2 B. 4 C. 8 D. 16 E. NOTA

22. Let $a = re^{i\theta}$ be a non-real value such that $f(a) = 0$. Which of the following expressions must be equal to $\sin\theta$?

A. $\frac{r\ln(2)}{2\theta}$ B. $\frac{2\theta}{r\ln(2)}$ C. $\frac{2r}{\theta\ln(2)}$ D. $\frac{\theta\ln(2)}{2r}$ E. NOTA

23. Which of the following is not true about $f(x)$?

A. $\lim_{x\to\infty} f(x) = \infty$ B. $\lim_{x\to-\infty} f(x) = -\infty$

C. $f(x)$ has complex roots D. $f(x)$ has a negative real root E. NOTA

24. Find the number of non-real solutions to $(x-1)(x^3 - 4x^2 + 10x - 20) = 0$.

A. 0 B. 1 C. 2 D. 3 E. NOTA

25. Solve for x if $x = \frac{4}{1+\frac{x}{1+\frac{x}{\cdots}}}$, and x is real.

A. 2 B. $\frac{4}{3}$ C. 1 D. $\frac{2}{3}$ E. NOTA

26. Which of the following is always true, when defined for $z \in \mathbb{C}, n \in \mathbb{R}$. Consider only the principal value of square roots and logarithms.

I. $\sqrt{z} \cdot \sqrt{z} = \sqrt{z^2}$. II. $\ln(z^n) = n\ln(z)$. III. $\ln(n^z) = z\ln(n)$

A. I, II, III B. I, III C. II, III D. III E. NOTA

27.
$$4x + 5y + 6z = 7 + i$$
$$7x + 4y + 5z = 6 + i$$
$$5x + 6y + 7z = 4 + i$$
If x, y, and z are complex numbers, find $12x + 10y + 12z$

A. $10 + 2i$ B. $11 + 2i$ C. $13 + 2i$ D. $7 + 5i$ E. NOTA

28. Consider the function $f(x) = Ax^4 + Bix^3 + Cx^2 + Dix + E$, where A, B, C, D, E are all real numbers. Given that $2 + i$ and $1 + 3i$ are roots of $f(x)$, find $-\frac{B}{A}$.

A. -6 B. 6 C. $-8i$ D. $8i$ E. NOTA

29. Which of the following ordered pairs (a, b) make $\big||z| - |z + a|\big| = b$ a non-degenerate hyperbola?

A. $(-2,0)$ B. $(-3,2)$ C. $(3,4)$ D. $(0,0)$ E. NOTA

30. For some real number x,
$$\cos(x) + \frac{\cos(2x)}{2} + \frac{\cos(3x)}{4} + \cdots + \frac{\cos(nx)}{2^{n-1}} + \cdots = \frac{1}{2}$$
Let
$$S = \sin(x) + \frac{\sin(2x)}{2} + \frac{\sin(3x)}{4} \ldots + \frac{\sin(nx)}{2^{n-1}} + \cdots$$
Given that $S > 0$ and $S = \frac{\sqrt{p}}{q}$, where $p, q \in \mathbb{Z}^+$ and $p + q$ is minimized. Find $p + q$.

A. 9 B. 10 C. 11 D. 12 E. NOTA