For each of the following questions, answer choice E. NOTA means none of the above answers is correct. In general, take brackets [] to mean a matrix, and vertical lines || to mean a determinant.

1. Find the shortest distance between the two lines $\frac{x-3}{2}=\frac{y+5}{3}=\frac{z-1}{6}$ and $\frac{x-4}{2}=\frac{y+1}{3}=\frac{z-6}{6}$.

A. 1 B. $\frac{\sqrt{122}}{7}$ C. $\frac{6\sqrt{5}}{5}$ D. $\sqrt{42}$ E. NOTA

2. Let A be a square matrix. If $|A|=3$, and $|4A|=192$, Find $|(2A)^{-1}|$.

A. $\frac{1}{6}$ B. $\frac{1}{12}$ C. $\frac{1}{24}$ D. $\frac{8}{3}$ E. NOTA

3. Find the rank of the following matrix:

$$\begin{bmatrix}4 & -3 & 6\\ 1 & 2 & 5\\ 6 & 1 & 16\end{bmatrix}$$

A. 0 B. 1 C. 2 D. 3 E. NOTA

4. Which of the following is not in row echelon form?

A. $\begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 1\end{bmatrix}$ B. $\begin{bmatrix}1 & 0\\ 0 & 3\\ 0 & 0\end{bmatrix}$ C. $\begin{bmatrix}4 & 2\\ 0 & 3\\ 0 & 0\end{bmatrix}$ D. $\begin{bmatrix}0 & 2\\ 0 & 0\\ 0 & 0\end{bmatrix}$ E. NOTA

5. Find the third row, second column of the inverse of the following matrix:

$$\begin{bmatrix}1 & 2 & -3 & 1\\ 2 & 3 & -6 & 2\\ 3 & -5 & -5 & 5\\ -2 & 5 & 5 & -5\end{bmatrix}$$

A. $\frac{3}{2}$ B. 0 C. $-\frac{3}{10}$ D. $-\frac{3}{2}$ E. NOTA

For questions 6-8, consider the following matrix:

$$\begin{bmatrix} 4 & -3 & 5 \\ 2 & 6 & 0 \\ 3 & 4 & 2 \end{bmatrix}$$

6. Find the determinant of the matrix.

A. 10 B. 15 C. -94 D. -10 E. NOTA

7. Find the sum of the entries of the first row in the inverse of the matrix.

A. 0 B. $\frac{9}{10}$ C. $\frac{3}{5}$ D. $\frac{4}{5}$ E. NOTA

8. If the eigenvalues of the matrix are λ_1, λ_2, and λ_3, find $\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$.

A. -35 B. -12 C. 12 D. 35 E. NOTA

9. Find the volume of the parallelepiped with edges defined by the vectors $\langle -1,2,5\rangle$, $\langle 3,-6,2\rangle, \langle 24,17,0\rangle$.

A. $\frac{1105}{6}$ B. 512 C. 432 D. 568 E. NOTA

10. Let the measure of the acute angle formed from the intersection of the lines $\frac{x-5}{4} = \frac{y-3}{4} = \frac{z-4}{7}$ and $\frac{x-14}{9} = \frac{y-9}{6} = \frac{z-6}{2}$ be θ. Find $\cos\theta$.

A. $\frac{46}{99}$ B. $\frac{74}{99}$ C. $\frac{11}{2}$ D. No intersection E. NOTA

11. The entries of a 2×2 matrix are all positive integers. given that the sum of its entries is 16, find the maximum value of the determinant.

A. 0 B. 40 C. 48 D. 64 E. NOTA

12. How many of the following are true?
    I. A nilpotent matrix must have a determinant of 0.
    II. An idempotent matrix must have a determinant of 1.
    III. A 2×2 matrix M can only be nilpotent if M^2 is the zero matrix.
    IV. The trace of an idempotent matrix is always equal to its rank.
    A. 1 B. 2 C. 3 D. 4 E. NOTA

13. Given that $4x + 4y + 7z = 18$, find the minimum value of $x^2 + y^2 + z^2$.
    A. $\frac{32}{9}$ B. 4 C. 5 D. 6 E. NOTA

14. Evaluate
$$\begin{bmatrix} \sqrt{6}+\sqrt{2} & -\sqrt{6}+\sqrt{2} \\ \sqrt{6}-\sqrt{2} & \sqrt{6}+\sqrt{2} \end{bmatrix}^{100} \cdot \begin{bmatrix} -\frac{1}{4} & \frac{\sqrt{3}}{4} \\ \frac{-\sqrt{3}}{4} & -\frac{1}{4} \end{bmatrix}^{200}.$$
    A. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ B. $\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$ C. $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ D. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ E. NOTA

15. A quadrilateral with vertices at $(0,0), (1,2), (6,4)$, and $(3,-2)$ is transformed using the matrix $\begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$. Find the area of the rotated quadrilateral.
    A. 4 B. 16 C. 32 D. 64 E. NOTA

16. Find the shortest distance between the point $(w, x, y, z) = (1, 4, 6, 1)$ and the space $2w + 4x - 5y + 6z = 13$ in hyperspace.
    A. 3 B. $\frac{19}{9}$ C. 2 D. $\frac{7}{9}$ E. NOTA

17. Find the resulting vector when $4i + 8j$ is rotated $\frac{\pi}{2}$ clockwise.

A. $4i - 8j$ B. $-8i + 4j$ C. $8i - 4j$ D. $-4i + 8j$ E. NOTA

For questions 18-20, consider the following information.

The Hill Cipher was used frequently to send confidential information in the military. It involves an encryption matrix that, when multiplied with letters taken two at a time, can encrypt a message. The letters A to Z are represented by the remainder when an entry in the resulting matrix is divided by 26, from 0 to 25, respectively. For example, using the encryption matrix $\begin{bmatrix}3 & 5\\7 & 9\end{bmatrix}$, if we wanted to encrypt the word "QI", we would get $\begin{bmatrix}3 & 5\\7 & 9\end{bmatrix}\begin{bmatrix}Q\\I\end{bmatrix} = \begin{bmatrix}3 & 5\\7 & 9\end{bmatrix}\begin{bmatrix}16\\8\end{bmatrix} = \begin{bmatrix}88\\184\end{bmatrix} = \begin{bmatrix}10\\2\end{bmatrix} = KC$. If there are an odd number of letters in the message being encoded, fill the last slot with a Z, or 25. Here is a conversion table for your convenience.

A	B	C	D	E	F	G	H	I	J	K	L	M
0	1	2	3	4	5	6	7	8	9	10	11	12

N	O	P	Q	R	S	T	U	V	W	X	Y	Z
13	14	15	16	17	18	19	20	21	22	23	24	25

18. Encrypt the message "ILOVEYOU" using the hill cipher with encryption matrix $\begin{bmatrix}1 & 2\\4 & 2\end{bmatrix}$. What is the fourth letter of the encryption?

A. D B. E C. I D. U E. NOTA

19. If the encryption matrix of a cipher is $\begin{bmatrix}1 & 2\\4 & 2\end{bmatrix}$, find the determinant of the decryption matrix.

A. $-\frac{1}{6}$ B. 1 C. -6 D. DNE E. NOTA

20. Kejin wanted to send Alan a secret message, so he encrypted it using a hill cipher with the encryption matrix $\begin{bmatrix}2 & 5\\1 & 3\end{bmatrix}$. He ended up sending the letters WFMTEJ. What is the topic of this message?

A. Love B. Math C. Sports D. Food E. NOTA

For questions 21-26, consider matrix $M = \begin{bmatrix} 2 & 4 & 1 \\ -1 & 1 & -1 \\ 2 & 4 & 0 \end{bmatrix}$.

21. Find the trace of M.

A. 3 B. 4 C. 7 D. 0 E. NOTA

22. Find the sum of the coefficients of the characteristic polynomial of M.

A. -12 B. 0 C. 8 D. 12 E. NOTA

23. Calculate the product of the eigenvalues of the matrix.

A. -6 B. 0 C. 4 D. 16 E. NOTA

24. If there exists a matrix K with $|K| > 0$ such that $M \cdot K = K$, find the determinant of K. If K does not exist, simply bubble D.

A. -6 B. 0 C. 4 D. K Does not exist E. NOTA

25. Given the equation $M \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 5 \\ 16 \end{bmatrix}$, find $x + y + z$.

A. 0 B. 1 C. 2 D. 3 E. NOTA

26. Find the determinant of the cofactor matrix of M.

A. -6 B. $-\frac{1}{6}$ C. 0 D. 36 E. NOTA

27. Construct all 2×2 matrices where all elements are from the set $\{0, 1\}$. Pick a random matrix from them, with equal probability. Call this matrix A. What is the probability that $A^2 = A$?

A. $\frac{1}{16}$ B. $\frac{1}{4}$ C. $\frac{1}{2}$ D. $\frac{5}{8}$ E. NOTA

28. Jeremy is very lazy and did not do his homework on January 1st, 2022. Assume that homework is given every day, once a day. The probability Jeremy does his homework on a given day, given that he did not do his homework the previous day, is 0.3, while the probability Jeremy does his homework on a given day, given that he did do his homework the previous day, is 0.5. What is the probability, rounded to the nearest thousandth, that Jeremy does his homework on January 1st, 2062?

A. 0.333 B. 0.375 C. 0.417 D. 0.444 E. NOTA

29. Samuel and Chris start at the same point at time $t = 0$. Chris moves along the path defined as $x(t) = t, y(t) = 2t - 1, z(t) = 4t + 2$, while Samuel moves along the path defined as $x(t) = 3t, y(t) = 4t - 1, z(t) = 2$. Calculate the distance between the two when they are the same distance away from the origin again at time $t > 0$.

A. 0 B. $2\sqrt{6}$ C. $\sqrt{590}$ D. $10\sqrt{6}$ E. NOTA

30. Find the determinant of the following:

$$\begin{vmatrix} 5 & 3 & 1 & 3 \\ 4 & 7 & 3 & 0 \\ 6 & 6 & 2 & 2 \\ 3 & 7 & 9 & 5 \end{vmatrix}$$

A. 7 B. 14 C. 28 D. 42 E. NOTA