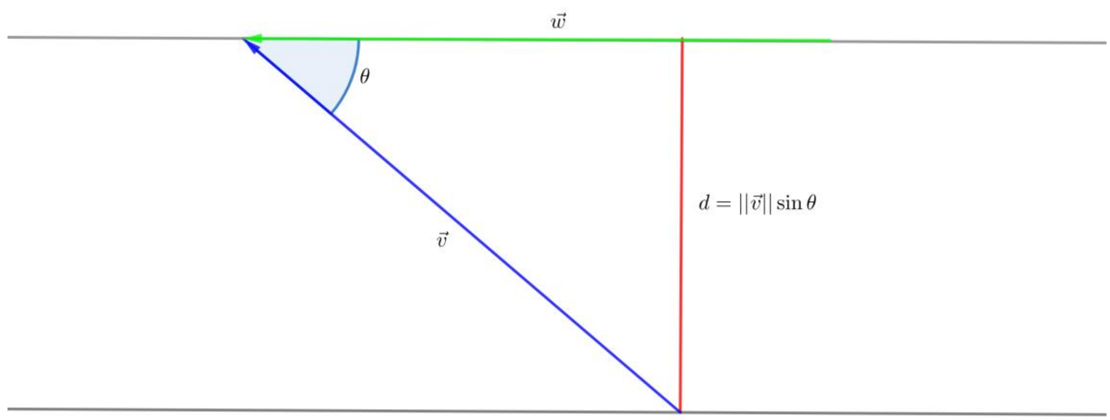
1. B Note that the two lines are parallel. Now, consider two parallel lines in space.



We are looking for d, the length of the line segment perpendicular to both lines with endpoints at each line. Notice that $d = ||\vec{v}|| \sin\theta$, where θ is the angle between the vector formed in the direction of the line and $\vec{v}$, the vector formed by a point on one line and a point on the other line. Remember that $\sin\theta = \frac{||\vec{v}\times\vec{w}||}{||\vec{v}||||\vec{w}||}$. Then, $d = ||\vec{v}|| \frac{||\vec{v}\times\vec{w}||}{||\vec{v}||||\vec{w}||} = \frac{||\vec{v}\times\vec{w}||}{||\vec{w}||}$. In our equations, $||\vec{v}||$ can be found by mapping the two points where $\frac{x-3}{2} = \frac{y+5}{3} = \frac{z-1}{6} = 0$, $\frac{x-4}{2} = \frac{y+1}{3} = \frac{z-6}{6} = 0$, which makes $< 1, 4, 5 >$. $||\vec{w}||$, the "slope" of the lines, is represented by the coefficients of t in the parametric form of the lines, which in this case is $< 2, 3, 6 >$. Solving for the cross product and the magnitude of $\vec{w}$ yields $\frac{\sqrt{122}}{7}$.

2. C If $|A| = 3$, and $|4A| = 192$, we multiply the determinant by a factor of $4^3 = 64$. This indicates that A is a 3×3 matrix. That means that when A is multiplied by 2, the determinant increases by a factor of $2^3 = 8$. The determinant of $|2A|$ is then 24. The determinant of the inverse of a matrix is just the reciprocal of the determinant of the matrix, leading to an answer of $\frac{1}{24}$.

3. C The rank is defined as the number of linearly independent rows of a matrix. Notice that the first row plus two times the second row is the third row. Since the first and second row cannot cancel each other out, this means that the rank of our matrix is **2.**

4. A There are two conditions for the row echelon form of a matrix: The zero rows must be at the bottom and the first nonzero entry of any row must not have any nonzero entries below it in its column. Answer choice **A** does not satisfy the second part of the requirements.

5. D First, calculate the value of the cofactor matrix at the second row, third column. Our determinant is

$$-\begin{vmatrix} 1 & 2 & 1 \\ 3 & -5 & 5 \\ -2 & 5 & -5 \end{vmatrix}.$$

Notice that the second row and the third row have the second and third columns in common. We can add the third row to the second row:

$$-\begin{vmatrix} 1 & 2 & 1 \\ 1 & 0 & 0 \\ -2 & 5 & -5 \end{vmatrix}.$$

Using minors, our determinant is

$$-(-1)\cdot(2\cdot(-5)-5\cdot 1)=-15.$$

Finally, divide by the determinant of our matrix. Multiply the first row of the matrix by 2 and subtract it from the second row. Additionally, add the fourth row to the third row:

$$\begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -2 & 5 & 5 & -5 \end{bmatrix}.$$

Using minors on the second row, we get

$$-1\begin{vmatrix} 1 & -3 & 1 \\ 1 & 0 & 0 \\ -2 & 5 & -5 \end{vmatrix}.$$

Minors one more time:

$$-(-1)\begin{vmatrix} -3 & 1 \\ 5 & -5 \end{vmatrix} = 15-5=10.$$

Our final answer is $-\frac{15}{10}=-\boldsymbol{\frac{3}{2}}$.

6. A Use the standard minor operation on the second row:

$$\begin{vmatrix} 4 & -3 & 5 \\ 2 & 6 & 0 \\ 3 & 4 & 2 \end{vmatrix} = -2\begin{vmatrix} -3 & 5 \\ 4 & 2 \end{vmatrix} + 6\begin{vmatrix} 4 & 5 \\ 3 & 2 \end{vmatrix} - 0\begin{vmatrix} 4 & -3 \\ 3 & 4 \end{vmatrix} = -2(-26)+6(-7)=\mathbf{10}.$$

7. D First, we find the determinant, which is 10 from the previous question. Now, find the sum of the entries of the first column of the cofactor matrix:

$$\sum e_1 = 6\cdot 2-4\cdot 0-(-3\cdot 2-5\cdot 4)+(-3)\cdot 0-5\cdot 6=8.$$

Divide by the determinant to get $\frac{8}{10}=\boldsymbol{\frac{4}{5}}$.

8. D Find the characteristic polynomial, done by subtracting x from each entry in the matrix's diagonal, and solving for the determinant:

$$\lambda(x)=\begin{vmatrix} 4-x & -3 & 5 \\ 2 & 6-x & 0 \\ 3 & 4 & 2-x \end{vmatrix} = -x^3+12x^2-35x+10.$$

The roots of our characteristic polynomial are the eigenvalues of our matrix. Solving for the sum taken two at a time is just an application of Vieta's formulas:

$$\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3=\frac{c}{a}=-\frac{35}{-1}=\mathbf{35}.$$

9. E To find the volume, simply take the triple scalar product:

$$V = \begin{vmatrix} -1 & 2 & 5 \\ 3 & -6 & 2 \\ 12 & -13 & 12 \end{vmatrix} = \mathbf{1105}.$$

10. B We first check to see if there is an intersection. (5,3,4) works for both lines, which means that we can safely find θ. To find theta, we note that

$$\cos\theta = \frac{u \cdot v}{||u||||v||}.$$

$u =< 4,4,7 >$ and $v =< 9,6,2 >$. Solving for $\cos\theta$, we get

$$\cos\theta = \frac{36 + 24 + 14}{9 \cdot 11} = \frac{\mathbf{74}}{\mathbf{99}}.$$

11. C We want to maximize $ac - bd$, given $a + b + c + d = 16$, for positive integers a, b, c, d. We want to maximize ac, and minimize bd. bd is minimized when $b = d = 1$, which is conveniently also when ac is maximized. given that $a + c = 14$, ac is maximized when $a = c = 7$. Our determinant is $7 \cdot 7 - 1 \cdot 1 = \mathbf{48}$.

12. C I. is true. This is one of the definitions of a nilpotent matrix.
II. is false. An idempotent matrix can have a determinant of 0. An example would be

$$\begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix}.$$

III. is true. This is because the trace of any power of a nilpotent matrix must be 0.
IV. is true. This is one of the definitions of an idempotent matrix.
Therefore, there are **3** correct statements.

13. B The Cauchy-Schwarz inequality tells us that

$$(4x + 4y + 7z)^2 \leq (x^2 + y^2 + z^2) \cdot (4^2 + 4^2 + 7^2).$$

This means that

$$18^2 \leq (x^2 + y^2 + z^2) \cdot 81;$$
$$x^2 + y^2 + z^2 \geq 4.$$

Equality occurs when $\frac{x}{4} = \frac{y}{4} = \frac{z}{7}$, which is obviously achievable. Our answer is **4.**

14. C Define $cis(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Notice that $cis^n(\theta) = cis(n\theta)$. We will use this to our advantage.
The first matrix is $4cis\left(\frac{\pi}{12}\right)$. The second matrix is $\frac{1}{2}cis\left(\frac{-2\pi}{3}\right)$. This means that our expression becomes

$$\left(4cis\left(\frac{\pi}{12}\right)\right)^{100} \cdot \left(\frac{1}{2}cis\left(-\frac{2\pi}{3}\right)\right)^{200} = cis\left(\frac{100\pi}{12}\right) \cdot cis\left(-\frac{400\pi}{3}\right) = cis(-\pi).$$

Our answer is then $\begin{bmatrix} \cos(-\pi) & -\sin(-\pi) \\ \sin(-\pi) & \cos(-\pi) \end{bmatrix} = \begin{bmatrix} \mathbf{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} \end{bmatrix}$.

15. C We can multiply the area of the original quadrilateral by the determinant of the rotation matrix. We use the shoelace formula to find the area of the quadrilateral. Line the points in clockwise order and perform the appropriate steps:

$$\begin{matrix} 0 & 1 & 6 & 3 \\ 0 & 2 & 4 & -2 \end{matrix}$$

$$\frac{|0 + 4 - 12 + 0 - 0 - 12 - 12|}{2} = 16.$$

The determinant of the rotation matrix is 2, which means that our final answer is $16 \cdot 2 = \mathbf{32}$.

16. B The shortest distance is defined as

$$\frac{|Aa + Bb + Cc + Dd - E|}{\sqrt{A^2 + B^2 + C^2 + D^2}},$$

Where the point (a, b, c, d) and the space A$w + Bx + Cy + Dz = E$ are being considered. Applying this formula,

$$\frac{|2 \cdot 1 + 4 \cdot 4 - 5 \cdot 6 + 6 \cdot 1 - 13|}{\sqrt{2^2 + 4^2 + 5^2 + 6^2}} = \mathbf{\frac{19}{9}}.$$

17. C Draw a good picture. We know that for the vector $< a, b >$, $4a + 8b = 0$ and $a^2 + b^2 = 4^2 + 8^2$. Our two options are <8, -4> and <-8, 4>. Since we are rotating clockwise, the vector should be in the 4th quadrant, giving us an answer of **<8, -4>**.

18. D Multiply the matrix $\begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$ by $\begin{bmatrix} 14 \\ 21 \end{bmatrix}$ to get $\begin{bmatrix} 4 \\ 20 \end{bmatrix}$. We are interested in the second row. 20 corresponds to **U** in the alphabet.

19. A The decryption matrix is just the inverse of the encryption matrix. To reverse an encryption, we need to multiply the left side by the inverse of the encryption matrix. The determinant of the inverse of our matrix is just the reciprocal of the determinant, which is $2 - 8 = -6$, giving us an answer of $-\mathbf{\frac{1}{6}}$.

20. D Take the inverse of the matrix. that gets us $\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$. Multiply this matrix by pairs of letters in the encryption. We get our decryption code is POTATO, which is clearly a **food**.

21. A The trace is the sum of the diagonal entries; $2 + 1 + 0 = \mathbf{3}$.

22. A The characteristic polynomial is calculated by subtracting x in each of the entries in the diagonal and finding the determinant. We know that $f(1)$ is the sum of the coefficients of a polynomial $f(x)$, which means we can subtract 1 from the diagonal and find its determinant.

$$\begin{vmatrix} 2-1 & 4 & 1 \\ -1 & 1-1 & -1 \\ 2 & 4 & 0-1 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 1 \\ -1 & 0 & -1 \\ 2 & 4 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 1 \\ -1 & 0 & -1 \\ 1 & 0 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 1 \\ 0 & 0 & -3 \\ 1 & 0 & -2 \end{vmatrix}$$

$$= 3 \cdot \begin{vmatrix} 1 & 4 \\ 1 & 0 \end{vmatrix} = \mathbf{-12}.$$

23. A The product of the eigenvalues is defined as the determinant of a matrix; We just need to find the determinant.

$$\begin{vmatrix} 2 & 4 & 1 \\ -1 & 1 & -1 \\ 2 & 4 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & -1 \\ 2 & 4 & 0 \end{vmatrix} = 1 \cdot \begin{vmatrix} -1 & 1 \\ 2 & 4 \end{vmatrix} = \mathbf{-6}.$$

24. D Since the determinant of M is not 1, and because $|K| > 0$, there is no such K that satisfies this.

25. B Subtracting the third row from the first row, it is easy to see that $z = -4$. Now, all that is remaining is a system of two equations and two variables,

$$2x + 4y = 16$$
$$-x + y = 1.$$

Solving for x and y gets $x = 2, y = 3$. $x + y + z = 2 + 3 - 4 = \mathbf{1}$.

26. D We can solve for the determinant of the cofactor matrix for a generic $n \times n$ matrix $|M|$. First, we know that the cofactor matrix transposed is the adjoint matrix, which is then multiplied by the reciprocal of the determinant, resulting in the inverse of M, which has determinant $\frac{1}{|M|}$. Let the determinant of the cofactor matrix be $|C|$. The transpose does not change the determinant, while multiplying by $\frac{1}{|M|}$ results in the multiplication of the determinant by $|M|^{-n}$. Because $|M|^{-n}|C| = |M|^{-1}$, $|C| = |M|^{n-1} = (-6)^2 = \mathbf{36}$.

27. C There are only 16 possible cases, so it would not be too bad to list all the cases out, but there are many ways of simplifying this question. First, look at the expression $A^2 = A$. Multiplying both by A^{-1} on the left side, assuming A is invertible, yields

$A^{-1} \times A \times A = A^{-1} \times A$, or $A = I$. This means that the only non-singular matrix that satisfies this is the identity matrix (this is also a property of idempotent matrices). Now, there are not a lot of extra cases to consider because our determinant must be equal to 0. We have the zero matrix, which trivially satisfies this property. then, we have the 4 matrices with one 1 and zeros for the rest of its entries. The only two that work here are when the 1 is on the diagonal, as the 1 needs to be multiplied to itself to produce 1 in the desired row and column. The only remaining case is where there is exactly one zero in each of the diagonal entries and the non-diagonal entries. There are 4 cases here, and by simple checking, and utilizing symmetry, all 4 of these satisfy our equation. That brings our total to 1+1+2+4=8. The total number of matrices that we could have chosen is $2^4 = 16$, which means that our answer is $\frac{8}{16} = \frac{1}{2}$.

28. B 2062 is a long way away from 2022 in terms of days, so to the nearest thousandth, we can assume that it will be an infinitely long period of time. After a long period of time, we assume that the probability of Jeremy doing homework the next day is fixed. In other words, we need to find an x such that $\begin{bmatrix} 0.5 & 0.3 \\ 0.5 & 0.7 \end{bmatrix}\begin{bmatrix} x \\ 1-x \end{bmatrix} = \begin{bmatrix} x \\ 1-x \end{bmatrix}$. Multiplying this matrix out, we get $0.5x + 0.3(1-x) = x$ and $0.5x + 0.7(1-x) = 1-x$, which are the same equation. Solving one of these, we get that $x = \frac{3}{8}$. We are asked to round this answer to the nearest thousandth. 2062 is so far off 2022 that our answer is going to be extremely close to $\frac{3}{8}$, or **0.375**.

29. D The time t for which the magnitudes of the two paths are the same should be considered.

$$t^2 + (2t-1)^2 + (4t+2)^2 = (3t)^2 + (4t-1)^2 + 4.$$

Solving this equation gets us $t = 0, 5$. We take $t = 5$, Getting us the points $(5,9,22)$ and $(15,19,2)$. The distance between them is calculated as $\sqrt{10^2 + 10^2 + 20^2} = \mathbf{10\sqrt{6}}$.

30. E Notice that the sum of the first and third columns is equal to the sum of the second and fourth. This means that the determinant is **0.**