Answer Key:

1. 55
2. 1
3. $3e^e$
4. e
5. DNE
6. $\frac{49}{20}$
7. $\frac{2\pi}{3} + \sqrt{3}$
8. 225
9. $\ln 3$
10. $2 \arctan \sqrt{x} + C$
11. 12500
12. $\frac{1}{5}$
13. 30
14. 1
15. $\frac{\sqrt{5}}{2} e^4$
16. π
17. ∞
18. e^e
19. $\frac{252\pi}{5}$
20. $\frac{3\pi}{2} + 2\sqrt{2}$
21. $\frac{\pi^2}{6}$
22. True
23. $\frac{3}{5}$
24. 10922
25. 1

Solutions:

1. We can compute $f'(x) = \sum_{i=0}^{9}(i+1)x^i$, and if we plug in $x = 1$, we get that $f'(1) = 55$.

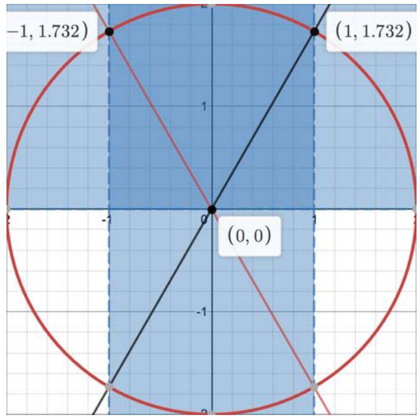
2. Note that we can move the n in the denominator into the log to get a more familiar limit:
$$\lim_{n\to\infty} \frac{1}{\ln\left(\left(1+\frac{1}{n}\right)^n\right)} = \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} \ln\left(\left(1+\frac{1}{n}\right)^n\right)} = \frac{1}{\ln\left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right)} = \frac{1}{\ln(e)} = 1$$

3. First, rewrite $x^{x\ln x}$ as $e^{x(\ln x)^2}$, then we compute the derivative with product rule, so $f'(x) = x^{x\ln x} \cdot \left((\ln x)^2 + \frac{2}{\ln x}\right)$, so $f'(e) = 3e^e$.

4. Note that if this limit exists, then the natural log this limit also exists, and vice versa. Thus, we can find the following limit with L'Hospital, and reverse the transformation:
$$\lim_{n\to\infty}\left(e^n - e^{\frac{n}{2}}\right)^{\frac{1}{n}}$$
$$= \lim_{n\to\infty} \frac{\ln\left(e^n - e^{\frac{n}{2}}\right)}{n}$$
$$= \lim_{n\to\infty} \frac{\frac{e^n - \frac{e^{\frac{n}{2}}}{2}}{e^n - e^{\frac{n}{2}}}}{1} = 1$$
Thus, the original limit is equal to e.

5. If we plug in $\frac{\pi}{2}$ into the given equation, we notice that we get 0^∞ which is indeterminate. Thus, we are motivated to take the log of the expression. However, note that if we approach $\frac{\pi}{2}$ from the right side, then $\cos n$ will be negative, thus we aren't able to take the log. This means that the answer is DNE.
6. We can compute $\int_0^1 f(x)dx = \sum_{i=1}^{6}\frac{1}{i}x^i = 2 + \frac{9}{20} = \frac{49}{20}$.

7. This integral represents the area of the intersection of the two blue regions in the circle:



We can break this region down into a $60°$ sector and two $30-60-90$ triangles, and so we get that the area is equal to $\frac{1}{6} \cdot 4\pi + 2 \cdot \frac{1 \cdot \sqrt{3}}{2} = \frac{2\pi}{3} + \sqrt{3}$.

8. We can compute the right-hand Riemann sum as

$$1 \cdot f(1) + 1 \cdot f(2) + \cdots + 1 \cdot f(5)$$
$$= \sum_{i=1}^{5} i^3 = 15^2 = 225.$$

9. We are motivated to use $\sin^2 x + \cos^2 x = 1$ to possibly split the fraction. Thus, we get the following:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\sin x \cos x} dx$$
$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin^2 x + \cos^2 x}{\sin x \cos x} dx$$
$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin^2 x}{\sin x \cos x} dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos^2 x}{\sin x \cos x} dx$$
$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x}{\cos x} dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos x}{\sin x} dx$$
$$= \ln|\sin x| - \ln|\cos x| \, \Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$
$$= \ln 3$$

10. We start with a u-substitution of $u = \sqrt{x}$ and $du = \frac{1}{2\sqrt{x}} dx$. Thus, we have

$$\int \frac{1}{x\sqrt{x} + \sqrt{x}} dx$$
$$= \int \frac{2}{u^2 + 1} du$$
$$= 2 \arctan u + C$$
$$= 2 \arctan \sqrt{x} + C.$$

11. We can model this as an optimization problem of trying to maximize the rectangle with side lengths x and $1000 - 2x$. This is maximized when $x = 250$, so the maximum area is 12500.

12. We have that $\frac{dV}{dt} = 5$ and $V = s^2 \cdot h$ when the height is $2\ cm$. Thus, we have $5 = s^2 \cdot \frac{dh}{dt}$ with $h = 2$, so we can compute $\frac{dh}{dt} = \frac{5}{25} = \frac{1}{5}$.

13. Since Newton's law of cooling applies to this situation, we have that $\frac{dT}{dt} = k(T - 80)$. Thus, we can solve to get $T = Ce^{kt} + 80$. When $t = 0, T = 200$, we have $C = 120$. When $t = 10, T = 140$, we have $k = \frac{1}{10} \ln \frac{1}{2}$. Thus, we wish to solve for t when $T = 95$, which occurs when $t = 30$.

14. We can represent this Riemann sum as a definite integral of the function $f(x) = xe^x$ over the interval $[0, 1]$. Taking the integral and using integration by parts, we get that it evaluates to $xe^x - e^x|_0^1 = 1$.

15. First, note that when the radius approaches 0, the angle approaches $-\infty$. Thus, we want to find the arc length from when the angles spans $-\infty$ to 2. Thus, we have

$$\int_{-\infty}^{2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$
$$= \int_{-\infty}^{2} \sqrt{e^{4\theta} + 4e^{4\theta}}\, d\theta$$
$$= \int_{-\infty}^{2} \sqrt{5} e^{2\theta} d\theta$$
$$= \frac{\sqrt{5}}{2} e^{2\theta}|_{-\infty}^{2} = \frac{\sqrt{5}}{2} e^4.$$

16. This is better known as the volume of Gabriel's horn, which can be computed as

$$V = \pi \int_1^\infty \frac{1}{x^2} dx = \pi.$$

17. This is better known as the surface area of Gabriel's horn, which is infinity. This can be computed as

$$SA = 2\pi \int_1^\infty \frac{1}{x} \sqrt{1 + \left(\frac{-1}{x^2}\right)^2} \, dx > 2\pi \int_1^\infty \frac{1}{x} dx = \infty$$

More details can be found here for 15 and 16: https://en.wikipedia.org/wiki/Gabriel%27s_Horn

18. Note that e^{e^x} and $\ln(\ln(x))$ are both impossible to integrate (if you found a way to, then kudos to you). However, they are inverse functions of each other:

$$\ln\left(\ln\left(e^{e^x}\right)\right) = \ln(e^x) = x.$$

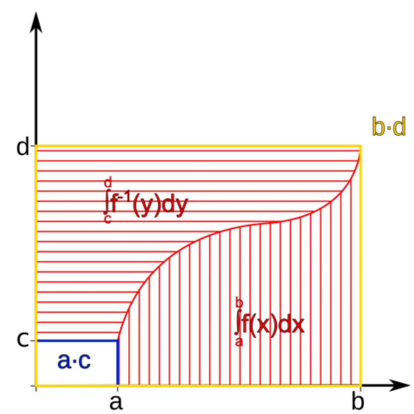
$$e^{e^{\ln(ln(x))}} = e^{\ln(x)} = x.$$

The last thing to note is that the bounds correspond to each other:

$$e^{e^0} = e, e^{e^1} = e^e,$$

$$\ln(\ln(e)) = 0, \ln(\ln(e^e)) = 1.$$

Thus, we can think of the area as follows:



Where $a = 0, b = 1, c = e, d = e^e$. Thus, we can compute the area to be

$$e^e \cdot 1 - e \cdot 0 = e^e.$$

19. The key idea behind this problem is to use Pappus' theorem to find the volume of the solid revolved around the line. Pappus' theorem states that the volume formed by rotating the region R around an axis is equal to the area of the region times the distance from the axis to the centroid of the region.

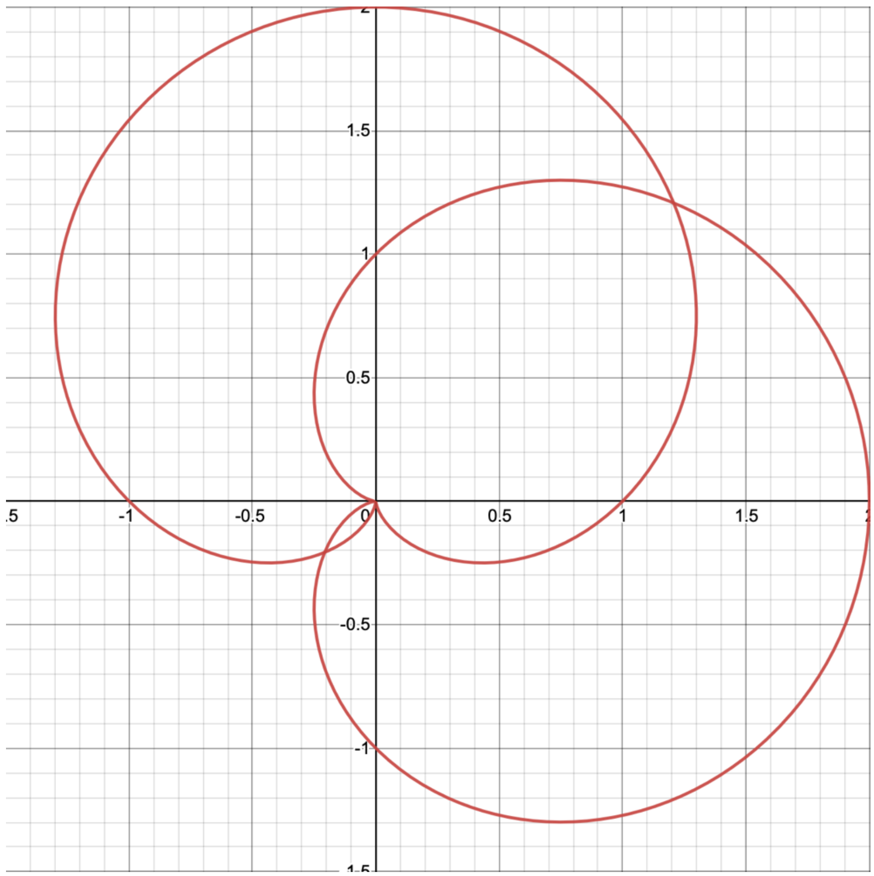
We can compute the area of the region to be $A = \frac{3 \cdot 6}{2} = 9.$

We can compute the centroid of the region to be $\left(\frac{0+3+3}{3}, \frac{0+0-6}{3}\right) = (2, -2)$

Thus, the distance from $(2, -2)$ to $3x + 4y = 12$ is $\left|\frac{2\cdot 3 - 2\cdot 4 - 12}{\sqrt{3^2+4^2}}\right| = \frac{14}{5}$

We can then compute the volume to be $2\pi \cdot \frac{14}{5} \cdot 9 = \frac{252\pi}{5}$.

20. Our picture is as follows:



Since we want the area in either limaçon, we can simplify computation by using symmetry. Note that if we find the area from $\frac{\pi}{4}$ to $\frac{5\pi}{4}$ of the top limaçon or $\frac{5\pi}{4}$ to $\frac{9\pi}{4}$ of the right limaçon, then we can simply double it to get the total area. Thus, we have that

$$A = 2 \cdot \frac{1}{2} \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (1 + \sin\theta)^2 d\theta$$

$$= \left(\theta - 2\cos\theta - \frac{\sin 2\theta}{4} + \frac{\theta}{2}\right)\Big|_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$$

$$= \frac{3\pi}{2} + 2\sqrt{2}.$$

21. Using the well-known identity that $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$, we conclude that the answer is $\frac{\pi^2}{6}$.

22. This is true. Since $\sum a_n$ converges, then that means there exists a value M such that $M > a_n$ for all $n \in \mathbb{N}$. Then $\sum a_n b_n < \sum M b_n = M \sum b_n$ which converges, so $\sum a_n b_n$ converges.
https://math.stackexchange.com/questions/85287/b-n-bounded-sum-a-n-converges-absolutely-then-sum-a-nb-n-also?rq=1 clarifies why the positive terms are necessary.

23. For this limit, we just have to plug in the power series for $\tan^{-1} x$ and then simplify. Thus, we have the following:

$$\lim_{x\to 0} \frac{3\tan^{-1} x^2 - 3x^2 + x^6}{x^{10}}$$
$$= \lim_{x\to 0} \frac{3\left(x^2 - \frac{x^6}{3} + \frac{x^{10}}{5}\right) - 3x^2 + x^6}{x^{10}}$$
$$= \frac{3}{5}.$$

24. This question is about using roots of unity filter:
If we look at the function $f(x) = (1+x)^{15} = \sum \binom{15}{i} x^i$, then we note that the coefficients of this function will follow $\binom{15}{i}$. Thus, we only want the coefficients of the multiple of 3s, which we can find by computing $\frac{f(1)+f(\omega)+f(\omega^2)}{3}$ where ω represents the cubic root of unity $\left(cis\left(\frac{2\pi}{3}\right)\right)$.
First, note that $1 + \omega + \omega^2 = 0$ since these three numbers are the roots of $x^3 - 1 = 0$, and the sum of the roots is 0. We also have that $\omega^3 = 1$ by definition. Thus, we have that

$$\frac{\left(f(1) + f(\omega) + f(\omega^2)\right)}{3}$$
$$= \frac{\left(\sum \binom{15}{i} + \sum \binom{15}{i}\omega^i + \sum \binom{15}{i}\omega^{2i}\right)}{3}$$
$$= \frac{3 \cdot \sum \binom{15}{3i} + (1+\omega+\omega^2)\sum \binom{15}{3i+1} + (1+\omega+\omega^2)\sum \binom{15}{3i+2}}{3}$$
$$= \sum \binom{15}{3i}$$

Thus, we have

$$f(1) = (1+1)^{15} = 2^{15}$$
$$f(\omega) = (1+\omega)^{15} = (-\omega^2)^{15} = -1$$
$$f(\omega^2) = (1+\omega^2)^{15} = (-\omega)^{15} = -1$$

So we have $\sum\binom{15}{3i} = \frac{2^{15}-2}{3}$.

Alternatively, we can compute $\binom{15}{0} = \binom{15}{15} = 1, \binom{15}{3} = \binom{15}{12} = 455, \binom{15}{6} = \binom{15}{9} = 5005$, so summing this up, we get that the answer is $2 \cdot (1 + 455 + 5005) = 2 \cdot 5461 = 10922$

25. We can compute this to be $0 + 1 = 1$.