MAΘ National Convention 2024 Geometry Hustle Solutions

25π	340π	48	$\frac{3}{4}$	216
$4\sqrt{3}-6$	$200+100\sqrt{2}$	$\frac{14}{\pi}$	1233	$32\pi+64$
$12\sqrt{15}$	16	$\frac{48}{5}$	$5\sqrt{3}$	$72-36\sqrt{3}$
63	$\frac{19\pi}{3}$	$20\sqrt{7}$	$\frac{75}{8}$	8
$\frac{1946}{3}$	46	38	84	$4+9\sqrt{5}$

1. 25π — 25π. The radius of this sphere is half the length of a side of the cube the sphere is inscribed in, or $\frac{5}{2}$. The surface area of a sphere is $4\pi r^2$, and plugging $\frac{5}{2}$ in for r gives 25π.

2. 340π — 340π. This is the sum of the volumes of two cones, one with height 12 and radius 5, and the other with height 5 and radius 12. The volume of a cone is $\frac{\pi r^2 h}{3}$, so the first has a volume of 100π, and the second has a volume of 240π. The sum of these volumes is 340π.

3. 48 — 48. The two possible triangles are those with side lengths $3x$, $3x$, and x^2-6, and $3x$, x^2-6, and x^2-6. The perimeter of the triangles is the sum of the lengths of its sides, which is equal to 32. Setting $3x+3x+x^2-6=32$ and $3x+x^2-6+x^2-6=32$, and solving, we get $x=-3\pm\sqrt{47}$ for the first equation and $x=4, \frac{-11}{2}$ for the second equation. Since x must be positive, the possible values of x are $\sqrt{47}-3$ and 4. The sum of these is $1+\sqrt{47}$, so $a+b=48$.

4. $\frac{3}{4}$ — $\frac{3}{4}$. Since these hexagons are both regular, they are similar, so the ratio of their areas is just the square of the ratio of their side lengths. If the smaller hexagon has side length 1, the larger hexagon will have side length $\frac{2\sqrt{3}}{3}$. The ratio of their side lengths is $1{:}\,\frac{2\sqrt{3}}{3}$, so the ratio of their side lengths is $1{:}\,\frac{4}{3}$, which is equal to $\frac{3}{4}$.

5. 216 — 216. The length of an internal tangent can be found by creating a right triangle with the internal tangent as one leg, the sum of the radii as the other, and the distance between the centers as the hypotenuse. The product of the internal tangents is the square of one internal tangent, which is $21^2-15^2=216$.

6. $4\sqrt{3}-6$ — $4\sqrt{3}-6$. The side length of this triangle is 8, which can be found given the area. The area of the triangle is also equal to the area of the three smaller triangles with bases 8 (sides of the triangle) and heights 2, 4, and x. In an equation this can be represented as $16\sqrt{3}=8\times 2/2+8\times 4/2+8x/2=4(2+4+x)$; solving for x gives $4\sqrt{3}-6$.

7. $200 + 100\sqrt{2}$ — $200 + 100\sqrt{2}$. The side length of this octagon is 10. $RPCL$ is a square and its area can be found by squaring one of its side lengths. One such side is RP, whose length can be found by extending TR and OP to form a right angle. This creates a 45°-45°-90° right triangle with hypotenuse $RO = 10$. The legs are thus each $5\sqrt{2}$. Using this, RP is also a hypotenuse of the right triangle with legs $5\sqrt{2}$ and $10 + 5\sqrt{2}$. The area of $RPCL$ is the same as $RP^2 = (5\sqrt{2})^2 + (10 + 5\sqrt{2})^2 = 200 + 100\sqrt{2}$.

8. $\frac{14}{\pi}$ — $\frac{14}{\pi}$. The perimeter of the heptagon is 28. The circumference of a circle is $2\pi r$, and setting this equal to 28 and solving for r yields $\frac{14}{\pi}$.

9. 1233 — 1233. The number of diagonals of a polygon is $\frac{n(n-3)}{2}$, and the sum of the interior angles of a polygon in degrees is $180(n-2)$, where n represents the number of sides for both expressions. Plugging in 9 for n, the number of diagonals is 27, and the sum of the interior angles is 1260. The positive difference between these two is 1233.

10. $32\pi + 64$ — $32\pi + 64$. The area of this shape is the area of the four circles minus the area of the four common sections (because they overlap, they will be double counted so they should be subtracted from the area of the circles). The area of each circle is $\pi(4)^2$, so the area of four circles is 64π. The area of the common sections is the double the area of a 90° sector of the circle with radius 4 minus the area of a 45°-45°-90° triangle with legs of length 4: $2\left(\frac{1}{4}\times\ \pi(4)^2 - \frac{1}{2}\times 4\times 4\right) = 8\pi - 16$. The total area of all common sections is four times this area, which is $32\pi - 64$. The area of the four leaf clover would then be $64\pi - (32\pi - 64) = 32\pi + 64$.

11. $12\sqrt{15}$ — $12\sqrt{15}$. The ratio of the areas is the square of the ratio of the side lengths. The ratio of the sides is thus $\frac{\sqrt{3}}{\sqrt{5}}$. The ratio of the volumes is the cube of the ratio of the sides: $\frac{3\sqrt{3}}{5\sqrt{5}}$. The ratio of the volumes is also the volume of the smaller icosahedron to 100, the larger icosahedron. Setting $\frac{3\sqrt{3}}{5\sqrt{5}} = \frac{x}{100}$ and solving for x yields $12\sqrt{15}$.

12. 16 — 16. The carrot farm is 16 miles north of the starting point. The barn is 36 miles east and 32 miles north of this point, which is given in the problem. The shortest distance would be the straight line distance given by reflecting one of these points over the river then finding the distance from one point to the other by Pythagorean theorem. This distance turns out to be 60 miles, which takes 12 hours to traverse if Ben walks at 5 miles per hour. The total number of hours he spends walking is this number plus the 4 hours he spends walking from the river to the carrot farm, which is 16 hours.

13. $\frac{48}{5}$ — $\frac{48}{5}$. Given the area of the trapezoid and the lengths of the bases, the height can be found by dividing the area, 40, by the sum of the bases, 20, and doubling this, which yields 4. Triangles SYA and DYN are similar in a ratio

of $\frac{8}{12}=\frac{2}{3}$, which means their heights are $\frac{2}{5}\times 4$ and $\frac{3}{5}\times 4$, respectively. Thus, their areas are $\frac{32}{5}$ and $\frac{72}{5}$, respectively, and subtracting this from the total area gives $\frac{96}{5}$, which is the sum of the areas of ANY and SDY. Because these two triangles are identical, they have the same area, so each has and area of $\frac{48}{5}$.

14. $5\sqrt{3}$ — $5\sqrt{3}$. Triangle OXS has side lengths 5, 7, and 8. The area of this triangle is $10\sqrt{3}$, which can be found via Heron's formula. This triangle also has a base of 8, and the height is half the length of XY. The height of this triangle is $\frac{5\sqrt{3}}{2}$, so $XY=5\sqrt{3}$.

15. $72-36\sqrt{3}$ — $72-36\sqrt{3}$. The side lengths of the square are 6, and so are the side lengths of the equilateral triangle. The height of this triangle is $3\sqrt{3}$, and so the perpendicular from Y to FI is $6-3\sqrt{3}$. FY^2 can be found using Pythagorean theorem: $3^2+(6-3\sqrt{3})^2=72-36\sqrt{3}$.

16. 63 — 63. Graphing and calculating areas or putting the points in order and shoestringing gives an area of 63.

17. $\frac{19\pi}{3}$ — $\frac{19\pi}{3}$. The area the iguana can roam is 240° sector of a circle with radius 3 feet and two 60° sectors with radius 1 foot. This total area is $\frac{2}{3}\times\pi(3)^2+2\times\frac{1}{6}\times\pi(1)^2=\frac{19\pi}{3}$.

18. $20\sqrt{7}$ — $20\sqrt{7}$. By law of cosines, $XR=\sqrt{30^2+20^2-2(30)(20)\cos 60^\circ}=10\sqrt{7}$. GR is just twice the length of XR, so it is $20\sqrt{7}$.

19. $\frac{75}{8}$ — $\frac{75}{8}$. The expected value of the dartboard is the sum of the points times the probability for each section. The probability of each section is its area divided by the total area of the dartboard. Thus, the expected score is $\frac{1}{16}\times 20+\frac{3}{16}\times 15+\frac{5}{16}\times 10+\frac{7}{16}\times 5=\frac{75}{8}$.

20. 8 — 8. The area of $PALY$ is the area of trapezoid $PALM$ minus the area of triangle MLY. The area of trapezoid $PALM$ is 12, as the bases and height are given. The area of triangle MLY is 4, as it has base 4 and height 2. The area of $PALY$ is thus 8.

21. $\frac{1946}{3}$ — $\frac{1946}{3}$. The volume of the frustum is the volume of the pyramid minus the volume of a smaller similar pyramid. The larger pyramid has base side length 10 and height $x+14$, while the smaller pyramid has base side length 3 and height x. They are similar, so $\frac{3}{10}=\frac{x}{x+14}$. $x=6$, so the volume of the larger pyramid is $\frac{1}{3}\times 10\times 10\times 20=\frac{2000}{3}$, and the volume of the smaller pyramid is $\frac{1}{3}\times 3\times 3\times 6=\frac{54}{3}$. The difference in these volumes is the volume of the frustum: $\frac{1946}{3}$.

22. 46 — 46. Calling the incenter Z, $m\angle\mathbf{DOZ}=m\angle\mathbf{ZOX}$ because they are the two angles split by an angle bisector. Because XY is parallel to OD, $m\angle\mathbf{OZX}=m\angle\mathbf{ZOX}$ as they are alternate interior angles. Thus, OZX is an isosceles

triangle, and sides ZX and OX can both be given a length a. With similar reasoning, DZY is isosceles, and sides ZY and DY can both be given length b. CX is then $17-b$, and OC is $29-a$, while XY is $a+b$. The perimeter of triangle CXY is the sum of these side lengths, which gives $17-b+29-a+a+b=46$.

23.	38	38. The kite's area is equal to half the product of its diagonals and given one diagonal of length 14 and an area of 168, the length of the other diagonal can be determined as 24. It is split into 12 and 12 by the diagonal of length 14. The quadrilateral whose vertices are the midpoints of the kite's sides is a rectangle with side lengths 12 and 7. These lengths can be found via similar right triangles. The total perimeter is twice the sum of the side lengths, which is 38.
24.	84	84. The semiperimeter of this triangle is $\frac{10+17+21}{2}=24$. By Heron's, the area of the triangle is $\sqrt{24\cdot 14\cdot 7\cdot 3}=\sqrt{48\cdot 3\cdot 7\cdot 7}=\sqrt{144\cdot 49}=12\cdot 7=84$.
25.	$4+9\sqrt{5}$	$4+9\sqrt{5}$. The centers of the circles are at $(3,-2)$ and $(-9,-8)$, which have a distance of $\sqrt{12^2+6^2}=\sqrt{180}=6\sqrt{5}$ between them. The radii of the circles are 4 and $3\sqrt{5}$. The distance between A and B will be maximized when they are placed where the line formed by extending the segment connecting the circles' centers intersects the circles again, one radius away from the respective center. The sum of the radii is $4+3\sqrt{5}$, which when added to the distance between the centers gives a maximum distance of $4+9\sqrt{5}$.