Answer Key:

1. 55
2. 50
3. 5040
4. 48
5. 17
6. $\frac{25}{12}$
7. 17
8. 4
9. -80
10. 2
11. 0,0,0
12. 25
13. 7
14. 3
15. 10
16. 2
17. $4\sqrt{3}$
18. $6\sqrt{6}$
19. 29
20. 100
21. 900
22. 1260
23. 11
24. 45
25. 120
26. 676
27. $\frac{1}{15}$
28. $-\frac{17}{108}$
29. $\frac{1}{6}$
30. $\frac{1}{2}$
31. 6
32. 2
33. 2
34. $3 \cdot 3 \cdot 3 - 3$
35. $5 \cdot 5 - \frac{5}{5}$
36. $F\ S$
37. $F\ W$
38. $S\ S$
39. $S\ U$
40. $N\ P$

Solutions:

1. Sum of first 10 positive integers is $\frac{10\cdot 11}{2} = 55$.

2. Every pair sum to -1, so since there are 49 pairs, we have $99 - 49 = 50$.

3. Multiplying first 7 positive integers is equal to 5040.

4. An easy way to think about this is to notice that this is the harmonic average of 40 and 60, which you might know as 48. Alternatively, solving it isn't that bad – you get $\frac{2\cdot 40\cdot 60}{40+60} = 48$.

5. We have $\sum_{i=0}^{5} \sqrt{2^i} = 7 + 7\sqrt{2} = 7\left(1 + \sqrt{2}\right)$. If we look at $\sqrt{2}$, we can approximate it as to be around $\frac{\sqrt{200}}{10}$ which is around 1.4. Thus, we have $7 \cdot (2.4) \approx 17$.

6. Sum of reciprocals is negative second to last coefficient divided by last coefficient, which is $\frac{-(-50)}{24} = \frac{25}{12}$.

7. We have $4^{\log_2 3} = 3^{\log_2 4} = 3^2 = 9$. Similarly, we can find $27^{\log_3 2} = 2^3 = 8$, so summing these two together gives 17.

8. Evaluating the determinant gives $0 \cdot 0 \cdot 0 + 1 \cdot 1 \cdot 2 + 2 \cdot 1 \cdot 1 = 4$.

9. We have that $f(0) = 0$, so we know the answer is of some form $f(x) = ax^2 + bx$. Next, we see that it is symmetrical about 0, so we have that $f(x) = ax^2$. Plugging in $x = 1$, we can compute $a = -5$. Thus, $f(4) = -5 \cdot 16 = -80$.

10. The formula for this is $\frac{r}{(1-r)^2}$ where r is the geometric ratio. You can derive this by multiplying what was given by r and subtracting it, and then repeating it one more time to get a constant.

11. Since we're free to choose any values for x, y, z. We see that $x = y = z = 0$ is a valid solution. If we want to prove that there are no other solutions, we can simply take the Diophantine equation mod 4, and note that the sum of any two squares cannot be 3 mod 4.

12. By Chinese Remainder Theorem, we know there exists a unique solution less than $7 \cdot 11 = 77$. If we look at possible solutions, we can test $3, 14, 25, 36, 47, 58, 69$, and we see that 25 satisfies 4 mod 7.

13. We can factor out a 7! to get $7!\,(1+8+72) = 7! \cdot 9^2$. Thus, the greatest prime factor is 7.

14. We can simply calculate the units digit and sum them up: $1^1 + 2^2 + 3^3 + 4^4 + 5^5 = 1 + 4 + 7 + 6 + 5 = 3$.

15. This will be some casework, thankfully there are very few cases. First, note that when $x = y = z$, then the only valid solution is when all of them are equal to 3. Thus, we only need to check what happens when $x = 2$. If $x = 2$, then we have $\frac{1}{y} + \frac{1}{z} = \frac{1}{2}$. We can expand and use Simon's favorite factoring trick to find that the other possible solutions are $y = 3, z = 6$ and $y = 4, z = 4$. Thus, there are $3! + \frac{3!}{2} + \frac{3!}{3!} = 10$ different assignments of the variables.

16. We have $\pi r^2 = 2\pi r$, so $r = 2$.

17. We have $\frac{s^2\sqrt{3}}{4} = 3s$, so $s = 4\sqrt{3}$.

18. We have $\frac{s\sqrt{6}}{3} \cdot \frac{A}{3} = 4A$, so $s = 6\sqrt{6}$.

19. We compute $20^2 + 21^2 = 400 + 441 = 841 = 29^2$.

20. We need two horizontal edges and two vertical edges, thus we have $\binom{5}{2} \cdot \binom{5}{2} = 100$.

21. We only need to consider three digits to choose because the remaining two will be fixed, so we have $9 \cdot 10 \cdot 10 = 900$.

22. There are 7 letters in $JEFFREY$. There is a duplicate F and a duplicate E, so when we consider arrangements, we will be double counting twice for each letter since we can switch their places interchangeably. Thus, we have $\frac{7!}{2! \cdot 2!} = 1260$.

23. We can compute the constant term as $x^2 \cdot \left(\frac{1}{x}\right)^2$ or 2^3. There are $\frac{3!}{2!}$ ways to choose the first one, and there are $\frac{3!}{3!}$ ways to choose the second one, giving a sum of $3 + 8 = 11$.

24. If we have that the sum of the hundred's and one's digit is equal to the ten's digit, then that means for middle digit i, there are i possible 3-digit numbers (hundred's digit can go from 1 to i). Thus, we have $1 + 2 + \cdots + 9 = 45$.

25. This is a classic stars and bars question. Since Konwoo wants at least one cookie, we'll remove one cookie and give it to him. For the remaining 7 cookies, we need to partition it into four groups, which requires 3 bars. Thus we are essentially looking for how many ways to compute $\binom{3+7}{3} = 120$.

26. There are 26 red cards to choose from for the first card, and 26 black cards to choose from for the second card, so there are a possible $26 \cdot 26 = 676$ possible combinations.

27. There are 16 total ways to flip four coins. Since we know at least one of them flipped heads, that means we can remove the way where we got four tails. We have only one way to get all heads, so the answer is $\frac{1}{15}$.

28. The probability that Mr. Lu rolls at least one 1 is equal to $1 - P(no\ ones) = 1 - \left(\frac{5}{6}\right)^3 = \frac{91}{216}$. Thus, his expected profits will be $\frac{91}{216} \cdot 1 + \frac{125}{216} \cdot (-1) = -\frac{34}{216} = -\frac{17}{108}$.

29. If we think about it symmetrically, Mr. Lu's expected winnings will be the same as his expected losses when we don't tie. Thus, we only need to consider his profits when we're tied, which is $\frac{1}{6} \cdot 1 = \frac{1}{6}$.

30. We can effectively ignore whatever the sum of the first 2021 rolls were as no matter what the sum is, there is a $\frac{1}{2}$ chance that his 2022nd roll will make the sum even.

31. The formula for a perfect number is $2^{p-1} \cdot (2^p - 1)$. Essentially, this banks on $2^p - 1$ to being a prime number, so the sum of its factors is $2^p - 1 + 1 = 2^p$. We can compute the sum of the factors of 2^{p-1} to be $2^p - 1$, so that's how intuitively, the sum of the factors is equal to twice itself. Alternatively, we can just try small numbers, and note that $6 = 1 + 2 + 3$ (it's also the equal to $1 \cdot 2 \cdot 3$ which is pretty cool)

32. We have $\sqrt{2 + x} = x$, so we can solve $x = 2$.
33. We have $2^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots} = 2^1 = 2$.

34. The only solution is $3 \cdot 3 \cdot 3 - 3$.

35. The only solutions is $5 \cdot 5 - \frac{5}{5}$.

36. This sequence follows MONDAY, TUESDAY, WEDNESDAY, THURSDAY, so the answer would be F S (order is days of the week starting from Monday).

37. This sequence follows SUMMER, FALL, WINTER, SPRING, SUMMER so the answer would be F W (order is seasons starting with summer).

38. This sequence follows ONE TWO THREE FOUR FIVE, so the answer would be S S (order is the alphabetical representation of numbers).

39. This sequence follows MERCURY, VENUS, EARTH, MARS, JUPITER so the answer would be S U (order is planets based on distance to the sun).

40. This sequence follows DOLLAR, HALF-DOLLAR, QUARTER, DIME so the answer would be N P (order is U.S coins from most to least value).