Unless otherwise specified, the domain and range of functions are limited to the real numbers. "NOTA" stands for "None Of These Answers." Read all questions carefully. Good luck!

1. The height (in meters) of a section of a roller coaster is represented by the continuous function $h(x)$ on the interval $[0, 6]$, where $h(0) = 8$ and $h(6) = 68$. Matt is riding the coaster and concludes that at some value $a \in (0, 6)$, his height $h(a)$ will be equal to 10 m. This is an application of what theorem in calculus?

   A. Extreme Value Theorem B. Mean Value Theorem
   C. Fundamental Theorem of Calculus D. Intermediate Value Theorem E. NOTA

2. A circular Ferris wheel rotates at a constant angular velocity. The wheel has an inner ring of seats located a distance of 90 ft from the center and an outer ring of seats located a distance of 120 ft from the center. A person in an outer seat travels with a tangential velocity 10 ft/s faster than a person in an inner seat. How fast, in radians/s, is the wheel rotating?

   A. 3 B. $\frac{4}{3}$ C. $\frac{3}{10}$ D. $\frac{1}{3}$ E. NOTA

For problems 3 and 4, the elevation of an infinite road at a horizontal displacement of x miles from $x = 0$ is given by $E(x)$ for constants c and d, where

$$E(x) = \begin{cases} \frac{e^x - e^{-x}}{x}, & x < 0 \\ \sin(x + c) + d, & x \geq 0 \end{cases}.$$

3. Find $\lim_{x \to -\infty} E(x)$.

   A. 1 B. 0 C. $-\infty$ D. ∞ E. NOTA

4. Given that $E(x)$ is continuous and differentiable everywhere and that $0 \leq c \leq \pi$, find the product cd.

   A. $\frac{\pi}{2}$ B. π C. 0 D. No c, d exist E. NOTA

5. A 3-D printed cork is formed by revolving trapezoid $DBTA$, with vertices $D(0, 0)$, $B(0, 4)$, $T(6, 1)$, and $A(6, 0)$, around the x-axis. Find the cork's volume.

   A. 84π B. 72π C. 42π D. 24π E. NOTA

6. In an electric circuit, the voltage V across an inductor is related to the inductance L in Henrys (H) and current I through the inductor by the equation $V = L\frac{dI}{dt}$. Find the instantaneous voltage across an inductor with inductance 4 H and current $I(t) = \cos(\pi t)$ at $t = 4/3$.

A. $2\pi\sqrt{3}$ B. 2π C. $2\sqrt{3}$ D. 2 E. NOTA

7. On the k^{th} day of a space expedition lasting n total days, a rocket travels $\frac{1}{3k+8n}$ kilometers. Find the limit of the total distance in km traveled by the rocket as n approaches infinity.

A. 0 B. $\frac{1}{3}\ln\left(\frac{11}{8}\right)$
C. $\ln\left(\frac{11}{8}\right)$ D. $\frac{1}{8}\ln\left(\frac{11}{3}\right)$ E. NOTA

8. Two spherical planets centered at $(10, 2, 9)$ and $(-10, 2, -6)$ have radii 4 and 7, respectively. What is the minimum distance between the planets?

A. 1 B. $6\sqrt{14}$ C. 14 D. 24 E. NOTA

For problems 9 *and* 10, *Mr. Louvre has functions* $f(x) = e^x \sin x$, $g(x) = x \csc x$, *and* $h(x) = xe^{-x}$.

9. Find the value of

$$\frac{h\left(\frac{\pi}{2}\right)g'\left(\frac{\pi}{2}\right) - g\left(\frac{\pi}{2}\right)h'\left(\frac{\pi}{2}\right)}{\left(h\left(\frac{\pi}{2}\right)\right)^2}.$$

A. $-e^{-\frac{\pi}{2}}$ B. $-e^{-\frac{\pi}{2}}$ C. $-e^{\frac{\pi}{2}}$ D. $e^{\frac{\pi}{2}}$ E. NOTA

10. Mr. Louvre defines a new function $L(x)$, where

$$L(x) = f'(x)g'(x)h'(x) + \begin{vmatrix} f(x) & f(x) & f'(x) \\ g(x) & g'(x) & g(x) \\ h'(x) & h(x) & h(x) \end{vmatrix}.$$

Find $L'(1)$.

A. 0 B. 4 C. -5 D. -2 E. NOTA

11. In a first-order chemical reaction, the rate $-\frac{d[A]}{dt}$ of decrease in a reactant's concentration $[A]$ is directly proportional to $[A]$. Which of the following functions for $[A](t)$ can describe a first-order chemical reaction? Assume $-\frac{d[A]}{dt}$ is nonzero.

I) $[A](t) = 4 - \ln t$ II) $[A](t) = -\ln t$ III) $[A](t) = 2e^{-3t}$

IV) $[A](t) = 1 + e^{-t}$ V) $[A](t) = 1$

A. III only B. III, IV only C. I, II, V only D. II only E. NOTA

12. Distinct integers α and β are chosen randomly from the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Let A be the area under the line $y = x$ and above the x-axis from $x = \min(\alpha,\ \beta)$ to $x = \max(\alpha,\ \beta)$. Find the probability that A is an integer.

A. $\frac{61}{121}$ B. $\frac{50}{121}$ C. $\frac{9}{11}$ D. $\frac{5}{11}$ E. NOTA

13. Jake jetpacks in the positive x direction along the curve $f(x) = x^4 - 2x^3 - 12x^2 + 24$, beginning at the point $(0, 24)$. At what value of x does his path change concavity?

A. $\frac{3+\sqrt{51}}{2}$ B. $\frac{1}{2}$ C. 1 D. 2 E. NOTA

14. A bracelet is fashioned out of a sphere with diameter 6 cm by boring a circular hole of diameter 4 cm all the way through it, concentric with the sphere. Find the bracelet's volume.

A. $\frac{10\pi\sqrt{5}}{3}$ B. $\pi(36 - 8\sqrt{5})$ C. $\frac{256\pi\sqrt{2}}{3}$ D. 12π E. NOTA

15. Isaac wants to approximate the real root of $f(x) = \frac{1}{1+x^2} - 1$. His initial guess is $x_0 = \sqrt{2}$, and he uses Newton's method to generate his next guess x_1. Find x_1.

A. $\frac{2\sqrt{2}}{3}$ B. $\frac{4\sqrt{2}}{3}$ C. $-\frac{\sqrt{2}}{2}$ D. $\frac{5\sqrt{2}}{2}$ E. NOTA

16. A caravan of 10 distinct buses is traveling to a football game, including the drumline bus and the color guard bus. Let N be the number of orderings of the 10 buses which place both the drumline and the color guard in the front half of the caravan. Find $\frac{N}{8!}$.

A. $5/14$ B. 10 C. 20 D. 45 E. NOTA

17. Ricky and Mort are playing a game. They take turns rolling a fair octahedral (8-sided) die displaying the numbers 1 through 8 and multiplying a shared product P by $e^{\pi n i/4}$, where n is the result of the die roll and $i = \sqrt{-1}$. P begins at 1, and Ricky rolls first. The first player whose roll makes P equal to -1 wins the game. What is the probability that Mort will win?

A. $\frac{1}{2}$ B. $\frac{7}{15}$ C. $\frac{3}{7}$ D. $\frac{7}{64}$ E. NOTA

18. Sharay writes an unusual trombone solo with infinitely many notes. The nth note lasts for $\frac{1}{\sqrt{2n^2+1}}$ seconds, notes are played one at a time, and there is no space between notes. Which of the following are true regarding this piece of music?

I) The piece will end in finite time.
II) As n approaches infinity, the length of the nth note approaches 0.
III) A concert consisting of this piece played 2023 times back-to-back will end in finite time.

A. All are true B. I, II only C. I, III only D. II only E. NOTA

19. A horizontal line is drawn through the parabola $3y = x^2$ through its focus. Find the area of the parabolic sector bounded by the line and the parabola.

A. 12 B. 3 C. 2 D. $\frac{3}{2}$ E. NOTA

20. Newton's second law of motion states that $\sum \vec{\mathbf{F}} = m\vec{\mathbf{a}}$ (the sum of forces acting on an object is equal to the object's mass multiplied by its acceleration). Two forces $\vec{\mathbf{F}}_1$ and $\vec{\mathbf{F}}_2$ act on a 20 kg asteroid, with no other forces present. If $\vec{\mathbf{F}}_1 = \langle -10, 30, 0\rangle$ and $\vec{\mathbf{F}}_2 = \langle 50, 50, 80\rangle$, what is the magnitude of the asteroid's acceleration, in $\mathrm{m/s^2}$? All forces are in $\mathrm{kg \cdot m/s^2}$.

A. 9 B. 10 C. 11 D. 12 E. NOTA

21. Jeff can eat a quart of ramen in 20 minutes. Eating from the same bowl, Jeff and Bryan can eat a quart of ramen together in 12 minutes. In how many minutes can Bryan eat a quart of ramen by himself?

A. 4 B. 16 C. 30 D. 32 E. NOTA

22. Define R as the polar region enclosed by the line $\theta = 0$ and the curve $r = 8\sqrt{\cos 2\theta}$ for $0 \leq \theta \leq \frac{\pi}{4}$. The line $y = kx$ in Cartesian coordinates divides R into two regions of equal area. Find k.

A. $2 - \sqrt{3}$ B. $2 + \sqrt{3}$ C. $\sqrt{2} - 1$ D. $\dfrac{\sqrt{3}}{3}$ E. NOTA

23. Lauren's velocity $v(t)$ in a Quadratic Formula One race depends on her fuel level $f(t) = \frac{1}{t}$ and the volume of audience cheers $c(t) = \frac{1}{1+t^7}$, such that $v(t) = f(t) \cdot c(t)$. If her position at time $t = 1$ is 0, her position at $t = 2$ can be expressed as $\ln(k)$. Find k.

A. $\dfrac{1}{\sqrt[7]{2}}$ B. $\sqrt[7]{\dfrac{127}{128}}$ C. $\sqrt[7]{\dfrac{129}{64}}$ D. $\sqrt[7]{\dfrac{256}{129}}$ E. NOTA

24. Undercover Agent Nagoshi must correctly guess a rational root of a cubic polynomial $p(x)$ with unknown integer coefficients to solve a crime. She is told (correctly) that $p(0) = 2023$ and that $\lim_{x\to\infty} \frac{p(x)}{x^3} = k$. From this information, she deduces that $p(x)$ has 48 different possible rational roots. Which of the following could be the value of k which Agent Nagoshi was told?

A. 10 B. 13 C. 14 D. 30 E. NOTA

25. Andre is born on January 1, 1998. His cousin Andy is born exactly 10 years later on January 1, 2008. On January 1, 2023, at what rate is the ratio of Andy's age to Andre's age changing, in years^{-1}?

A. $-\dfrac{2}{45}$ B. 0 C. $\dfrac{2}{125}$ D. $\dfrac{8}{125}$ E. NOTA

26. A bowl of Fortunate Marshmallows cereal consists of n total pieces, out of which k are marshmallows. Let $P_{k,n}$ be the probability that Helene will draw a marshmallow every time on n random draws of a single piece, with replacement. Evaluate the limit:

$$\lim_{n\to\infty} \sum_{k=1}^{n} P_{k,n}$$

A. $\dfrac{e}{e-1}$ B. $\dfrac{1}{e-1}$ C. 1 D. $\dfrac{e+1}{e}$ E. NOTA

27. A tennis ball is to be launched from a cannon on Earth with an initial velocity of 20 m/s at an angle θ to the ground, for $0 < \theta \leq \pi/2$. The acceleration due to gravity is 10 m/s^2 and other effects are negligible. What value for θ should be chosen to maximize the greatest height reached by the ball during its flight?

A. $\pi/2$ B. $\pi/3$ C. $\pi/4$ D. $\pi/6$ E. NOTA

28. Daniel's elves have 10 hours to make the maximum possible number of toys. The elves can use the time either to produce toys or to produce decorative lights which motivate the elves and increase the production rate of toys (not of lights). The elves can produce 8 toys/hr and 2 lights/hr in the absence of decorative lights, and each light increases the toy production rate by $100(e-1)\%$. How many hours should the elves devote only to making toys, given that all lights are made before all toys? (Assume that fractional lights/toys are allowed.)

A. $\frac{1}{8}$ B. $\frac{1}{2}$ C. 1 D. 10 E. NOTA

29. Biff is studying mods in his number theory class. What is the remainder when the value of

$$\int_1^3 (\ln 3)^4 3^{\left(x+3^x+3^{3^x}+3^{3^{3^x}}\right)}\,dx$$

is divided by 10?

A. 7 B. 3 C. 2 D. 0 E. NOTA

30. Let $P(n) = \prod_{k=1}^{2023}(kn)$. Evaluate

$$\frac{\sum_{n=1}^{2023} P(n)}{\sum_{n=1}^{2023} P(2n)}.$$

A. $\frac{1}{2^{2023}}$ B. $\frac{1}{2^{(2023^2)}}$ C. $\frac{1}{2023!}$ D. $\frac{1}{2}$ E. NOTA