Answers:

1. D	7. B	13. D	19. D	25. C
2. D	8. C	14. E $\left(\frac{20\pi\sqrt{5}}{3}\right)$	20. E (6)	26. A
3. D	9. D	15. C	21. C	27. A
4. A	10. D	16. C	22. A	28. B
5. C	11. A	17. B	23. D	29. D
6. A	12. D	18. D	24. A	30. A

Solutions:

1. D The $\boxed{\text{Intermediate Value Theorem}}$ states that given a continuous function $f(x)$ on the interval $[a, b]$, for any value k in between $f(a)$ and $f(b)$, there exists some c in the interval (a, b) such that $f(c) = k$. Since $h(0) < 10 < h(6)$, then according to the Intermediate Value Theorem there exists some value $a \in (0, 6)$ such that $h(a) = 10$. (Note that if $h'(x)$, the rate of change of $h(x)$, had been the subject of the question, this would have instead been an application of the Mean Value Theorem.)

2. D The angular velocity ω of an object revolving with tangential velocity v about a point a distance of r away is given by the formula $v = r\omega$. (Derive this by differentiating both sides of the equation $x = r\theta$ with respect to time.) Since the angular velocity is constant everywhere on the wheel, the velocities v_1 and v_2 of an inner and an outer rider, respectively, are given by the equations $v_1 = 90\omega$ and $v_2 = 120\omega$. $v_2 - v_1 = 10 \rightarrow 120\omega - 90\omega = 10 \rightarrow 30\omega = 10 \rightarrow \omega = \boxed{1/3}$.

3. D
$$\lim_{x\to-\infty} E(x) = \lim_{x\to-\infty} \frac{e^x - e^{-x}}{x}$$
This is in the indeterminate form $\frac{\infty}{\infty}$.
By L'Hopital's Rule, the limit becomes
$$\lim_{x\to-\infty} \frac{e^x + e^{-x}}{1} = \lim_{x\to-\infty} e^x + \lim_{x\to-\infty} e^{-x}$$
The first limit approaches 0 while the other approaches infinity, so their sum approaches $\boxed{\infty}$.

4. A Since each piece of $E(x)$ is continuous and differentiable over its respective domain, the only point that needs to be considered is where $x = 0$.

For $E(x)$ to be continuous at $x = 0$, it must be true that
$$\lim_{x\to0^-} \frac{e^x - e^{-x}}{x} = \lim_{x\to0^+} \sin(x + c) + d$$
The first limit is in the indeterminate form $\frac{0}{0}$.
By L'Hopital's Rule, the limit becomes
$$\lim_{x\to0^-} \frac{e^x + e^{-x}}{1} = \lim_{x\to0^+} \sin(x + c) + d$$

$$\lim_{x\to 0^-} e^x + \lim_{x\to 0^-} e^{-x} = \lim_{x\to 0^+} \sin(x+c) + d$$
$$2 = \sin c + d$$

For $E(x)$ to be differentiable at $x = 0$, it must be true that

$$\lim_{x\to 0^-} \frac{d}{dx}\left(\frac{e^x - e^{-x}}{x}\right) = \lim_{x\to 0^+} \frac{d}{dx}(\sin(x+c)+d)$$

Using the quotient rule, $\frac{d}{dx}\left(\frac{e^x-e^{-x}}{x}\right) = \frac{x(e^x+e^{-x})-(e^x-e^{-x})}{x^2}$.

$$\lim_{x\to 0^-} \frac{x(e^x+e^{-x})-(e^x-e^{-x})}{x^2} = \lim_{x\to 0^+} \cos(x+c)$$

The first limit is in the indeterminate form $\frac{0}{0}$.

Using L'Hopital's Rule, the limit becomes

$$\lim_{x\to 0^-} \frac{(e^x - e^{-x})}{2} = \lim_{x\to 0^+} \cos(x+c)$$
$$0 = \cos c$$

$0 \le c \le \pi$, so $c = \frac{\pi}{2}$.

Using the first equation $2 = \sin c + d$,

$$d = 2 - \sin\frac{\pi}{2} = 2 - 1 = 1$$

$cd = \left(\frac{\pi}{2}\right)(1) = \boxed{\frac{\pi}{2}}$.

5. C <u>Calculus Method:</u>

The equation of the line containing vertices $B(0,4)$ and $T(6,1)$ is $y - 4 = \frac{1-4}{6-0}(x - 0) \rightarrow y - 4 = -\frac{1}{2}x \rightarrow y = 4 - \frac{1}{2}x$.

Using the washer method for the volume of a solid of rotation,

$$V = \pi\int_0^6 \left(4 - \frac{1}{2}x\right)^2 dx$$

Using the substitution $u = 4 - \frac{1}{2}x$, $du = -\frac{1}{2}dx$,

$$V = -2\pi\int_4^1 u^2 du = 2\pi\int_1^4 u^2 du = \frac{2\pi}{3}u^3\Big|_1^4 = \frac{2\pi}{3}(64-1) = \boxed{42\pi}$$

<u>Alternative Method:</u>

Note that the solid of revolution forms a frustrum, or a cone truncated parallel to its base. The volume of a frustrum can be calculated by finding the volume V_1 of the cone before truncation, finding the volume V_2 of the portion removed (also a cone), and subtracting the two.

The equation of the line containing vertices $B(0,4)$ and $T(6,1)$ is $y - 4 = \frac{1-4}{6-0}(x - 0) \rightarrow y - 4 = -\frac{1}{2}x$, so its x-intercept is $(-2)(-4) = 8$.

So, the cone with volume V_1 has radius 4 and height 8 and the cone with volume V_2 has radius 1 and height 2.

Then $V = V_1 - V_2 = \frac{\pi}{3}(4)^2(8) - \frac{\pi}{3}(1)^2(2) = \frac{\pi}{3}(128-2) = \boxed{42\pi}$

6. A $\frac{dI}{dt} = I'(t) = -\pi\sin(\pi t) \rightarrow I'\left(\frac{4}{3}\right) = -\pi\sin\left(\frac{4\pi}{3}\right) = -\pi\left(-\frac{\sqrt{3}}{2}\right) = \frac{\pi\sqrt{3}}{2}$

$$V = L\frac{dI}{dt} = (4)\left(\frac{\pi\sqrt{3}}{2}\right) = \boxed{2\pi\sqrt{3}}$$

7. B The total distance traveled by the rocket over the n-day expedition is the sum

$$\sum_{i=1}^{n}\frac{1}{3i+8n}$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{3i+8n} = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n\left(3\frac{i}{n}+8\right)} = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\cdot\frac{1}{3\left(\frac{i}{n}\right)+8}$$

This is a Riemann sum representing the integral

$$\int_0^1\frac{1}{3x+8}dx = \frac{1}{3}\ln|3x+8|\,\Big|_0^1 = \frac{1}{3}(\ln 11 - \ln 8) = \boxed{\frac{1}{3}\ln\left(\frac{11}{8}\right)}$$

8. C Minimizing the distance of a path connecting the spheres' centers will have the same effect, as every point on a sphere's surface is the same distance away from its center. The shortest path between the spheres' centers is a straight line. Thus, the shortest path from the surface of one sphere to another is along the line connecting the sphere's centers. The distance between the sphere's centers is

$$\sqrt{\left(10-(-10)\right)^2+(2-2)^2+\left(9-(-6)\right)^2} = \sqrt{400+0+225} = \sqrt{625} = 25$$

Every point on a sphere's surface is located a radius length away from the sphere's center. The spheres have radii 4 and 7, so the minimum distance between a point on each of the spheres' surfaces is $25-4-7=14$.

9. D Recognizing the form of the quotient rule,

$$\frac{h(x)g'(x)-g(x)h'(x)}{\left(h(x)\right)^2} = \left(\frac{g(x)}{h(x)}\right)' = \left(\frac{x\csc x}{xe^{-x}}\right)' = (e^x\csc x)'$$

$$= -e^x\csc x\cot x + e^x\csc x$$

$$\rightarrow -e^{\frac{\pi}{2}}\csc\frac{\pi}{2}\cot\frac{\pi}{2} + e^{\frac{\pi}{2}}\csc\frac{\pi}{2} = -e^{\frac{\pi}{2}}(1)(0) + e^{\frac{\pi}{2}}(1) = \boxed{e^{\frac{\pi}{2}}}$$

10. D

$$L(x) = f'g'h' + \begin{vmatrix} f & f & f' \\ g & g' & g \\ h' & h & h \end{vmatrix} = f'g'h' + fg'h + fgh' + f'gh - f'g'h' - 2fgh$$

$$= (fg'h + fgh' + f'gh) - 2fgh = (fgh)' - 2fgh$$

$$fgh = (e^x\sin x)(x\csc x)(xe^{-x}) = x^2$$

$$L(x) = (x^2)' - 2x^2 = 2x - 2x^2$$

$$L'(x) = 2 - 4x \rightarrow L'(1) = 2 - 4(1) = \boxed{-2}$$

11. A $-\frac{d[A]}{dt} = k[A] \rightarrow -\frac{1}{[A]}\, d[A] = k\, dt \rightarrow \int_{[A]_0}^{[A](t)} -\frac{1}{[A]}\, d[A] = \int_0^t k\, dt \rightarrow$
$-\ln[A]\Big|_{[A]_0}^{[A](t)} = kt\Big|_0^t \rightarrow -\ln([A](t)) + C_1 = kt \rightarrow \ln([A](t)) = C_1 - kt \rightarrow$
$[A](t) = Ce^{-kt}$
Note that $k \neq 0$ since $-\frac{d[A]}{dt}$ and $[A]$ must be in direct proportion and $\frac{d[A]}{dt} \neq 0$.
The only function in this form is thus $\boxed{\text{III) } [A](t) = 2e^{-3t}}$.

12. D Without loss of generality, let $\alpha < \beta$. Then $A = \int_\alpha^\beta x\, dx = \frac{x^2}{2}\Big|_\alpha^\beta = \frac{\beta^2-\alpha^2}{2} = \frac{(\beta+\alpha)(\beta-\alpha)}{2}$.
Thus, A is an integer when $(\beta+\alpha)(\beta-\alpha)$ is even.
There are four cases (where O=Odd and E=Even):

α	β	$\beta+\alpha$	$\beta-\alpha$	$(\beta+\alpha)(\beta-\alpha)$
O	O	E	E	E
O	E	O	O	O
E	O	O	O	O
E	E	E	E	E

Therefore, the only cases for which $(\beta+\alpha)(\beta-\alpha)$ is even are when α and β have the same parity.
There are 6 even numbers and 5 odd numbers in the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.
Thus, there is a $\frac{6}{11}$ probability that the first number chosen is even, and a $\frac{5}{10} = \frac{1}{2}$ probability that the second number chosen is even, so the probability that α and β are both even is $\left(\frac{6}{11}\right)\left(\frac{1}{2}\right) = \frac{3}{11}$.
Similarly, there is a $\frac{5}{11}$ probability that the first number chosen is odd, and a $\frac{4}{10} = \frac{2}{5}$ probability that the second number chosen is odd, so the probability that α and β are both odd is $\left(\frac{5}{11}\right)\left(\frac{2}{5}\right) = \frac{2}{11}$.
The final probability that A is an integer is $\frac{3}{11} + \frac{2}{11} = \boxed{\frac{5}{11}}$.

13. D

$$f(x) = x^4 - 2x^3 - 12x^2 + 24$$
$$f'(x) = 4x^3 - 6x^2 - 24x$$
$$f''(x) = 12x^2 - 12x - 24 = 12(x^2 - x - 2) = 12(x-2)(x+1)$$

A function's concavity changes where its second derivative changes sign. $f''(x)$ changes sign at $x = -1$ and $x = 2$. We are only considering the domain $[0, \infty)$, so Jake's path changes concavity only at $x = \boxed{2}$.

14. E This shape is equivalent to the shape created by revolving the area bounded by the semicircle $y = \sqrt{9 - x^2}$ and the line $y = 2$ around the x-axis.
$\sqrt{9 - x^2} = 2 \rightarrow x^2 = 9 - 4 \rightarrow x = \pm\sqrt{5}$
By the washer method,

$$V = \pi \int_{-\sqrt{5}}^{\sqrt{5}} \left(\sqrt{9 - x^2}\right)^2 - 2^2\, dx = \pi \int_{-\sqrt{5}}^{\sqrt{5}} 5 - x^2\, dx = 2\pi \int_0^{\sqrt{5}} 5 - x^2\, dx =$$

$$2\pi\left(5x-\frac{x^3}{3}\right)\Big|_0^{\sqrt{5}} = 2\pi\left(5\sqrt{5}-\frac{5\sqrt{5}}{3}\right) = 2\pi\left(\frac{2(5\sqrt{5})}{3}\right) = \boxed{\frac{20\pi\sqrt{5}}{3}}$$

15. C

$$f(x) = \frac{1}{1+x^2} - 1$$

$$f'(x) = (-1)(2x)(1+x^2)^{-2} = -\frac{2x}{(1+x^2)^2}$$

Newton's method uses the following recursive formula to generate each guess:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_0 = \sqrt{2}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = \sqrt{2} - \left(\frac{-\frac{2}{3}}{-\frac{2\sqrt{2}}{9}}\right) = \sqrt{2} - \frac{3}{\sqrt{2}} = \frac{2\sqrt{2}}{2} - \frac{3\sqrt{2}}{2} = \boxed{-\frac{\sqrt{2}}{2}}$$

16. C There are $\binom{5}{2} = 10$ ways to choose 2 places out of the front 5 for the drumline and color guard buses. Since we may switch the order of these two buses, we multiply this by 2 to get 20 possible orderings of the two buses in the front half of the caravan. The 8 remaining buses can be placed within the 8 remaining places in any order, so there are 8! orderings for these eight buses. This produces $20 \cdot 8!$ total orderings, so $\frac{N}{8!} = \boxed{20}$.

17. B Let P after m turns be expressed as $P = e^{\frac{\pi n_1 i}{4}} \cdot e^{\frac{\pi n_2 i}{4}} \cdot e^{\frac{\pi n_3 i}{4}} \cdot \ldots e^{\frac{\pi n_m i}{4}} = e^{\frac{\pi i}{4}(n_1+n_2+n_3+\cdots+n_m)}$. Since $e^{k\pi i} = -1$ only for odd integer values of k, then $S = n_1 + n_2 + n_3 + \cdots + n_m$, the sum of all die rolls up to the last, must leave a remainder of 4 when divided by 8.

For the first roll, $n = 4$ satisfies this, so on his first turn Ricky has a $1/8$ chance of winning and a $7/8$ chance of passing the die on to Mort. Note that for any positive integer value of S, S is always at most 8 away from a larger number that leaves a remainder of 4 when divided by 8. Thus, no matter where Ricky's first roll lands (besides 4), Mort has a $1/8$ chance of rolling the value of n that causes S to leave a remainder of 4 when divided by 8. So, Mort's chance of winning on his first turn is $\left(\frac{7}{8}\right)\left(\frac{1}{8}\right)$, and the chance that he will pass it back to Ricky is $\left(\frac{7}{8}\right)\left(\frac{7}{8}\right)$.

This sequence continues, where the chance of a player winning on the m^{th} roll is $\left(\frac{7}{8}\right)^{m-1}\left(\frac{1}{8}\right)$. Since Ricky rolls first, Mort's total probability of winning can be found by adding together all the terms of this sequence for which m is even:

$$\left(\frac{7}{8}\right)\left(\frac{1}{8}\right) + \left(\frac{7}{8}\right)^3\left(\frac{1}{8}\right) + \left(\frac{7}{8}\right)^5\left(\frac{1}{8}\right) + \cdots$$

This is an infinite geometric series with first term $\frac{7}{64}$ and common ratio $\frac{49}{64}$, so the sum can be computed as $\frac{\frac{7}{64}}{1-\frac{49}{64}} = \frac{\frac{7}{64}}{\frac{15}{64}} = \boxed{\frac{7}{15}}$.

18. D I) The total length of the piece is the sum of the durations of every note:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n^2+1}}$$

By the Limit Comparison Test,

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{2n^2+1}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{\sqrt{2n^2+1}} = \lim_{n\to\infty} \sqrt{\frac{n^2}{2n^2+1}}$$

$$= \lim_{n\to\infty} \sqrt{\frac{n^2+\frac{1}{2}}{2n^2+1} - \frac{\frac{1}{2}}{2n^2+1}} = \lim_{n\to\infty} \sqrt{\frac{1}{2} - \frac{\frac{1}{2}}{2n^2+1}} = \frac{1}{\sqrt{2}}$$

Since the limit as n approaches infinity is finite and positive, the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n^2+1}}$ and the series $\sum_{n=1}^{\infty} \frac{1}{n}$ must exhibit the same behavior. $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the p-series test, so $\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n^2+1}}$ also diverges. Thus, Sharay's solo does not end in finite time. False

II) $\lim_{n\to\infty} \frac{1}{\sqrt{2n^2+1}} = 0$. True

III) $2023 \sum_{n=1}^{\infty} \frac{1}{\sqrt{2n^2+1}} \geq \sum_{n=1}^{\infty} \frac{1}{\sqrt{2n^2+1}}$, so if $\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n^2+1}}$ diverges then $2023 \sum_{n=1}^{\infty} \frac{1}{\sqrt{2n^2+1}}$ must also diverge. Thus, the concert does not end in finite time. False

From these options, only $\boxed{\text{II}}$ is true.

19. D $3y = x^2$ is a parabola in the form $4p(y-k) = (x-h)^2$ with vertex $(h,k) = (0,0)$ and parameter $p = 3/4$. Recall that p is the distance from the vertex to the focus. Also, since the horizontal line drawn through the parabola passes through its focus, the portion of the line intercepted by the parabola is the latus rectum of the parabola, which has a length of $4p = 3$. The area of the inscribed parabolic sector is $4/3$ times the area of the inscribed triangle with the latus rectum as its base and the segment from the focus to the vertex as its height. This triangle has area $\frac{1}{2}(3)\left(\frac{3}{4}\right) = \frac{9}{8}$, so the parabolic sector has area $\frac{4}{3}\left(\frac{9}{8}\right) = \boxed{\frac{3}{2}}$.

20. E $\sum \vec{\mathbf{F}} = m\vec{\mathbf{a}} \rightarrow \langle -10, 30, 0\rangle + \langle 50, 50, 80\rangle = 20\vec{\mathbf{a}} \rightarrow \langle 40, 80, 80\rangle = 20\vec{\mathbf{a}} \rightarrow \vec{\mathbf{a}} = \langle 2, 4, 4\rangle \rightarrow |\vec{\mathbf{a}}| = \sqrt{2^2+4^2+4^2} = \sqrt{36} = \boxed{6}$

21. C Let J be Jeff's rate and B be Bryan's rate, both in quarts consumed per minute. Using the fact that rate = work/time,

$$J = \frac{1}{20}$$
$$J + B = \frac{1}{12}$$
$$B = \frac{1}{12} - J = \frac{1}{12} - \frac{1}{20} = \frac{5-3}{60} = \frac{2}{60} = \frac{1}{30}$$

Bryan's rate is $\frac{1}{30}$ quarts/minute, so it takes him $\boxed{30}$ minutes to eat a quart of ramen.

22. A The curve $r = 8\sqrt{\cos 2\theta}$ for $0 \le \theta \le \frac{\pi}{4}$ is the quarter of the lemniscate $r^2 = 64\cos 2\theta$ lying the first quadrant. The area of R can be found using polar integration:

$$\frac{1}{2}\int_0^{\frac{\pi}{4}} r^2\,d\theta = \frac{1}{2}\int_0^{\frac{\pi}{4}} 64\cos 2\theta\,d\theta = 16\sin 2\theta\,\Big|_0^{\frac{\pi}{4}} = 16\sin\frac{\pi}{2} - 16\sin 0 = 16$$

The line $y = kx$ on the Cartesian plane passes through the origin. This line may be expressed in polar coordinates as $\theta = \alpha$, where $\alpha = \tan^{-1} k$. If $\theta = \alpha$ divides R into two regions of equal area, then $\frac{1}{2}\int_0^{\alpha} r^2\,d\theta = \frac{1}{2}\int_\alpha^{\frac{\pi}{4}} r^2\,d\theta = \frac{1}{2}\left(\frac{1}{2}\int_0^{\frac{\pi}{4}} r^2\,d\theta\right) = \frac{1}{2}(16) = 8$. Focusing on the integral from 0 to α,

$$\frac{1}{2}\int_0^{\alpha} r^2\,d\theta == \frac{1}{2}\int_0^{\alpha} 64\cos 2\theta\,d\theta = 16\sin 2\theta\,\Big|_0^{\alpha} = 16\sin 2\alpha - 16\sin 0$$
$$= 16\sin 2\alpha$$

Setting this equal to 8 (half the area of R),

$16\sin 2\alpha = 8 \rightarrow \sin 2\alpha = \frac{1}{2} \rightarrow 2\alpha = \frac{\pi}{6} + 2\pi k,\ \frac{5\pi}{6} + 2\pi k \rightarrow \alpha = \frac{\pi}{12} + \pi k, \frac{5\pi}{12} + \pi k$

Since $0 \le \theta \le \frac{\pi}{4}$, the only possible value for α is $\frac{\pi}{12}$.

$$k = \tan\alpha = \tan\frac{\pi}{12} = \boxed{2 - \sqrt{3}}$$

(If $\tan\frac{\pi}{12}$ is not known, derive it using the tangent half-angle identity:

$\tan\frac{\pi}{12} = \frac{\sin\frac{\pi}{6}}{1+\cos\frac{\pi}{6}} = \frac{\frac{1}{2}}{1+\frac{\sqrt{3}}{2}} = \frac{1}{2+\sqrt{3}} = \frac{1(2-\sqrt{3})}{(2+\sqrt{3})(2-\sqrt{3})} = \frac{2-\sqrt{3}}{4-3} = 2 - \sqrt{3}$)

23. D

$$v(t) = f(t)\cdot c(t) = \frac{1}{t}\left(\frac{1}{1+t^7}\right) = \frac{1}{t+t^8}$$

For a time $T \ge 1$, her position x is given by

$$x(T) = x(1) + \int_1^T v(t)\,dt$$

$$x(2) = 0 + \int_1^2 \frac{1}{t+t^8}\,dt = \int_1^2 \frac{1}{t+t^8}\,dt = \int_1^2 \frac{\frac{1}{t^8}}{\frac{1}{t^7}+1}\,dt$$

This form sets up the substitution $u = \frac{1}{t^7} + 1$, $du = -\frac{7}{t^8}\,dt$:

$$\int_1^2 \frac{\frac{1}{t^8}}{\frac{1}{t^7}+1}\,dt = -\frac{1}{7}\int_2^{\frac{129}{128}} \frac{1}{u}\,du = \frac{1}{7}\int_{\frac{129}{128}}^{2} \frac{1}{u}\,du = \frac{1}{7}\ln|u|\,\Big|_{\frac{129}{128}}^{2} = \frac{1}{7}\left(\ln 2 - \ln\frac{129}{128}\right)$$

$$= \frac{1}{7}\ln\frac{256}{129} = \ln\sqrt[7]{\frac{256}{129}} = \ln k$$

$$k = \boxed{\sqrt[7]{\frac{256}{129}}}$$

24. A By the Rational Root Theorem, for a polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where P is the set of all divisors of a_0 and Q is the set of all divisors of a_n, every rational root of $f(x)$ is equal to or the negation of an element of P divided by an element of Q.

$$p(x) = ax^3 + bx^2 + cx + d$$
$$p(0) = d = 2023 = 7 \cdot 17^2$$
$$\lim_{x\to\infty} \frac{p(x)}{x^3} = \lim_{x\to\infty} \frac{ax^3 + bx^2 + cx + d}{x^3} = a = k$$

Using the sets P and Q as defined above,

$$P = \{1, 7, 17, 7\cdot 17, 17^2, 7\cdot 17^2\}$$
$$Q = \{\text{divisors of } k\}$$

If all elements of Q are relatively prime to all elements of P, then the number of different possible rational roots is simply the product of the number of elements in each, multiplied by 2 to account for both positive and negative values. P has 6 elements, so a value of k for which this is the case will have $2 \cdot 6 \cdot |Q| = 48 \rightarrow |Q| = 4$, where $|Q|$ is the number of elements in Q.

Looking first at the answer choices for which this is the case, out of 10, 13, and 30, only $\boxed{10}$ has exactly 4 divisors.

14 also has 4 divisors, but since some of these are not relatively prime to some of the divisors of 2023, there will be fewer than 48 possible rational roots.

25. C If t is the number of years after January 1, 2008, then the functions $f(t) = t$ and $g(t) = t + 10$ represent Andy's and Andre's ages, respectively. Then $r(t) = \frac{f(t)}{g(t)}$ represents the ratio of Andy's age to Andre's age t years after January 1, 2008.

$r(t) = \frac{t}{t+10} \rightarrow r'(t) = \frac{t+10-t}{(t+10)^2} = \frac{10}{(t+10)^2}$

On January 1, 2023, $t = 2023 - 2008 = 15$, so the rate of change of the ratio of their ages is $r'(15) = \frac{10}{(25)^2} = \frac{10}{625} = \boxed{\frac{2}{125}}$

26. A The probability of drawing a marshmallow one time is $\frac{k}{n}$, so the probability $P_{k,n}$ that a marshmallow is drawn every time on n draws with replacement is $\left(\frac{k}{n}\right)^n$.

$$\lim_{n\to\infty}\sum_{k=1}^{n}P_{k,n}=\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^n=\lim_{n\to\infty}\left(\frac{1}{n}\right)^n+\left(\frac{2}{n}\right)^n+\left(\frac{3}{n}\right)^n+\cdots+\left(\frac{n-1}{n}\right)^n+\left(\frac{n}{n}\right)^n$$

Note that the order of the summation may be reversed:

$$=\lim_{n\to\infty}\left(\frac{n}{n}\right)^n+\left(\frac{n-1}{n}\right)^n+\left(\frac{n-2}{n}\right)^n+\cdots+\left(\frac{2}{n}\right)^n+\left(\frac{1}{n}\right)^n$$

$$=\lim_{n\to\infty}\sum_{j=1}^{n}\left(\frac{n-(j-1)}{n}\right)^n=\lim_{n\to\infty}\sum_{j=1}^{n}\left(1-\frac{(j-1)}{n}\right)^n$$

Since $\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n=e^a$, then $\lim_{n\to\infty}\left(1+\frac{-(j-1)}{n}\right)^n=e^{-(j-1)}$, so the sum becomes

$$\lim_{n\to\infty}\sum_{j=1}^{n}e^{-(j-1)}=\lim_{n\to\infty}1+e^{-1}+e^{-2}+\cdots+e^{-(n-1)}=\frac{1}{1-e^{-1}}=\boxed{\frac{e}{e-1}}$$

27. A Velocity is a vector quantity whose components may be considered independently. The initial velocity is 20 m/s, so the vertical component is $20\sin\theta$ and the horizontal component is $20\cos\theta$. Since only the greatest height reached by the ball must be maximized, only the vertical component must be maximized. $\sin\theta$ is maximized when $\theta=\frac{\pi}{2}$, so $20\sin\theta$ is also maximized at $\boxed{\frac{\pi}{2}}$.

28. B Let x and y be the number of toys and lights produced, respectively, and t be the number of hours spent making toys. Since every light produced increases the 8 toy/hr production rate by $100(e-1)\%$ (or equivalently, multiplies the rate by e), we have the set of equations:

$$x=(8\cdot e^y)t$$
$$y=2(10-t)$$

Maximize the value of x by first rewriting x as a function of t and then setting its derivative with respect to t equal to 0:

$$x=\left(8\cdot e^{2(10-t)}\right)t=(8t)e^{2(10-t)}$$
$$\frac{dx}{dt}=(8t)(-2)e^{2(10-t)}+8e^{2(10-t)}=0$$
$$8e^{2(10-t)}(-2t+1)=0$$

$e^{2(10-t)}$ is never 0 for any real t, so it may be ignored.

$$-2t+1=0$$
$$t=\boxed{\frac{1}{2}}$$

One can check that this is a local maximum by noting that $-2t+1$ changes from positive to negative at $t=\frac{1}{2}$.

29. D $\int_1^3(\ln 3)^4 3^{\left(x+3^x+3^{3^x}+3^{3^{3^x}}\right)}\,dx=\int_1^3(\ln 3)^4(3^x)\left(3^{3^x}\right)\left(3^{3^{3^x}}\right)\left(3^{3^{3^{3^x}}}\right)\,dx$

Note that $\frac{d}{dx}(a^x) = (\ln a)a^x$ for any $a > 0$. Extending this to the "power tower" $a^{a^{a^{\cdots^{a^x}}}}$ with n a's, its derivative is $(\ln a)^n (a^x)\left(a^{a^x}\right)\left(a^{a^{a^x}}\right)\ldots\left(a^{a^{a^{\cdots^{a^x}}}}\right)$.

Therefore, $\int_1^3 (\ln 3)^4 (3^x)\left(3^{3^x}\right)\left(3^{3^{3^x}}\right)\left(3^{3^{3^{3^x}}}\right)\,dx = 3^{3^{3^{3^x}}}\Big|_1^3$

$$= 3^{3^{3^{3^3}}} - 3^{3^{3^3}}$$

The question asks for the remainder when $3^{3^{3^{3^3}}} - 3^{3^{3^3}}$ is divided by 10, which is simply the units digit of $3^{3^{3^{3^3}}} - 3^{3^{3^3}}$.

Looking first at $3^{3^{3^{3^3}}}$, let $b = 3^{3^{3^3}} \rightarrow 3^{3^{3^{3^3}}} = 3^b$. The units digit of powers of 3 follows the repeating four-term pattern $3, 9, 7, 1$ .... Thus, the remainder when b is divided by 4 will determine the units digit of 3^b. $b = 3^{3^{3^3}} = (4-1)^{3^{3^3}}$. The only term not divisible by 4 in the binomial expansion of $(4-1)^{3^{3^3}}$ is $(-1)^{3^{3^3}} = -1$, so b is 1 less than a multiple of 4 and thus leaves a remainder of 3 when divided by 4. Therefore, 3^b will end in a 7 (the third term in the repeating pattern).

Similarly, looking at the second term $3^{3^{3^3}} = 3^c$, $c = 3^{3^3} = (4-1)^{3^3} \rightarrow (-1)^{3^3} = -1$ is the only term not divisible by 4, so c also leaves a remainder of 3 when divided by 4, and 3^c will end in a 7 as well.

Then the units digit of $3^{3^{3^{3^3}}} - 3^{3^{3^3}}$ will be $7 - 7 = \boxed{0}$.

30. A

$$P(n) = \prod_{k=1}^{2023}(kn) = (n)(2n)(3n)\ldots(2023n) = 2023!\,n^{2023}$$

$$P(2n) = \prod_{k=1}^{2023}\big(k(2n)\big) = (2n)(2\cdot 2n)(3\cdot 2n)\ldots(2023\cdot 2n) = 2023!\,2^{2023}n^{2023}$$

$$\frac{\sum_{n=1}^{2023} P(n)}{\sum_{n=1}^{2023} P(2n)} = \frac{\sum_{n=1}^{2023} 2023!\,n^{2023}}{\sum_{n=1}^{2023} 2023!\,2^{2023}n^{2023}} = \frac{2023!\sum_{n=1}^{2023} n^{2023}}{2023!\,2^{2023}\sum_{n=1}^{2023} n^{2023}} = \boxed{\frac{1}{2^{2023}}}$$