All uppercase letter variables are positive integers unless otherwise stated. All fractions containing uppercase letter variables are in lowest terms. NOTA means "None of the Above."

~~~~~~ *Good luck and have fun!* ~~~~~~

1. Croix has set up her Roomba factory on the coordinate plane, since she knows that making shoelaces isn't that valuable. The octagonal factory has vertices at $(1,0)$, $(2,0)$, $(0,2)$, $(-2,0)$, $(0,-2)$, $(0,-1)$, $(-1,0)$, and $(0,1)$ in that order. Find the area of Croix's factory.

   A. $2\sqrt{2}$ B. 4 C. $\frac{9}{2}$ D. $2\sqrt{2}+2$ E. NOTA

2. Ursula takes advantage of the lack of shoelace producers at Luna Nova and sets up her convex quadrilateral factory with vertices at the points $(3,0)$, $(4,2)$, $(6,9)$, and $(2,5)$. Find the area of Ursula's factory.

   A. $\frac{27}{2}$ B. 14 C. $\frac{29}{2}$ D. 15 E. NOTA

3. Sucy uniformly randomly picks a point in the region defined by $1 \leq x^2 + y^2 \leq 100$ and makes it the center of a circle of radius 3 for her garden. If this circle completely lies in the region, then her mushroom garden will grow. Find the probability that Sucy is successful.

   A. $\frac{1}{3}$ B. $\frac{49}{99}$ C. $\frac{20}{33}$ D. $\frac{10}{11}$ E. NOTA

4. Akko and Diana are meeting at a coffee shoppe to go on a date. Both of them have lots of homework to do, so Akko can only stay at the shoppe for 40 minutes and Diana can only stay for 30 minutes. Both of them arrive at the shoppe independently at a uniformly random time between 3:00 PM and 5:00 PM today. If the probability that Akko and Diana are at the coffee shoppe at the same time and can go on a date equals $\frac{L}{W}$, find $L + W$.

   A. 11 B. 215 C. 217 D. 431 E. NOTA
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5. During a performance, Shiny Chariot's cape billows out behind her so that its projection over the first quadrant of the xy-plane is a rectangle with opposite corners at the origin and $\left(\frac{2\pi}{3}, \frac{\pi}{4}\right)$. Its height over this region at any given point is $1 + \frac{1}{4}\sin\left(4(x+y)\right)$. Find the volume of the region in space above the rectangle but below Shiny Chariot's cape.

A. $\frac{\pi^2}{6} - \frac{\sqrt{3}}{32}$ B. $\frac{\pi^2}{6} - \frac{\sqrt{3}}{64}$ C. $\frac{\pi^2}{6} + \frac{\sqrt{3}}{64}$ D. $\frac{\pi^2}{6} + \frac{\sqrt{3}}{32}$ E. NOTA

6. Ryuko is standing at the point $(-2,5,-1)$. Satsuki is standing somewhere on the plane $3x + 4y + 5z = 12$. Find the minimum possible distance between them.

A. $\frac{\sqrt{2}}{10}$ B. $\frac{\sqrt{2}}{5}$ C. $\frac{3\sqrt{2}}{10}$ D. $\frac{2\sqrt{2}}{5}$ E. NOTA

For questions 7-8, consider a frustum in space. Its larger base is in the xy-plane, centered at the origin, and has radius 4. Its smaller base is in the plane $z = 6$, centered at $(0, 0, 6)$, and has diameter 4.

7. Find the volume of this frustum.

A. 32π B. 48π C. 56π D. 64π E. NOTA

8. If the surface area of this frustum is $A\pi + B\pi\sqrt{C}$ for squarefree C, find $A + B + C$.

A. 38 B. 40 C. 42 D. 46 E. NOTA

9. If the radius of the largest sphere that can be inscribed in a cone with base radius 4 and height 12 is is $\frac{A}{B}\left(\sqrt{C} - D\right)$ for squarefree C, find $A + B + C + D$.

A. 18 B. 20 C. 21 D. 22 E. NOTA

10. The equation $3x^2 + 3y^2 - 8xy + 12x - 9y - 2023 = 0$ represents what non-degenerate conic section?

A. Circle B. Noncircular Ellipse C. Hyperbola D. Parabola E. NOTA

11. The region bounded by the graphs of $y = 2 + \sin(2x)$, $y = \frac{4x}{3\pi}$, and the y-axis is rotated over the x-axis. If the volume of the solid formed is equal to $A\pi + \frac{B\pi^2}{C}$, find $A + B + C$.

A. 35 B. 37 C. 43 D. 45 E. NOTA

12. Find the maximum area of an isosceles triangle inscribed in the ellipse $\frac{x^2}{4} + \frac{y^2}{2025} = 1$.

A. $45\sqrt{3}$ B. $\frac{135\sqrt{3}}{2}$ C. $90\sqrt{3}$ D. $\frac{225\sqrt{3}}{2}$ E. NOTA

13. After cutting four identical square pieces from its corners, Trevor folds a 6-foot by 9-foot rectangular piece of cardboard into a rectangular prism box to house all of his graduate school textbooks. In cubic feet, if the largest volume he can enclose is equal to $A + B\sqrt{C}$ for squarefree C, find $A + B + C$.

A. 19 B. 20 C. 21 D. 24 E. NOTA

14. A rectangle has sides parallel to the axes. Opposite vertices of the rectangle are at the origin and the point (a^3, b). The rectangle is rotated over the line $y = -2$ to form a solid. b is decreasing at a rate of 9 units per second. Given that the solid has a constant volume of 96π, find the positive rate of change of a when $a = 1$.

A. $\frac{1}{2}$ B. $\frac{5}{9}$ C. $\frac{5}{8}$ D. $\frac{2}{3}$ E. NOTA

15. A regular dodecagon with side length 2 is inscribed in a square whose side length is minimized. Find the area of the region inside the square but outside the dodecagon.

A. $1 + 4\sqrt{3}$ B. $4 + 4\sqrt{3}$ C. $4 + 8\sqrt{3}$ D. $8 + 8\sqrt{3}$ E. NOTA

16. Find the area of the region inside both $r = 1$ and $r = 1 + \cos\theta$.

A. $\frac{5\pi}{4} - 2$ B. $\frac{3\pi}{2} - 2$ C. $\frac{3\pi}{4} + 2$ D. $\frac{5\pi}{4} + 1$ E. NOTA

17. Sirius walks in a heart-shaped path along the curve $r = 5 + 5\cos\theta$. Let the distance he covers in one single complete circuit of this path be D. Find $\lfloor D \rfloor$.

A. 27 B. 31 C. 39 D. 40 E. NOTA

18. A curve is parametrically defined by $(x, y) = (2t - 4, 3t - t^2)$. Find the area bounded by this curve in the first quadrant.

A. $\frac{5}{6}$ B. $\frac{7}{3}$ C. $\frac{20}{3}$ D. 9 E. NOTA

19. Evaluate: $\lim_{n\to\infty} \int_0^{\pi n} e^{-x}|\sin x|\, dx$.

A. $\frac{e^{\pi}-1}{4(e^{\pi}+1)}$ B. $\frac{e^{\pi}+1}{4(e^{\pi}-1)}$ C. $\frac{e^{\pi}-1}{2(e^{\pi}+1)}$ D. $\frac{e^{\pi}+1}{2(e^{\pi}-1)}$ E. NOTA

For questions 20-21, you may refer to the following table of values for the twice-differentiable function y, whose first two derivatives are y' and y''.

x	-1	0	1	3	5	6
y	4	-2	7	-3	7	0
y'	-5	3	4	-2	0	9
y''	0	2	-2	-8	6	4

20. Approximate the value of $\int_{-1}^{5} xy\, dx$ using the Trapezoidal rule with 3 equal subintervals.

A. 15 B. 19 C. 27 D. 43 E. NOTA

21. Evaluate: $\int_1^6 (xy'' + y''y + (y')^2 + y')\, dx$.

A. -2 B. 22 C. 50 D. 118 E. NOTA

22. Circles of radius 1 are drawn with their centers at $(0,0)$, $(0,4)$, $(2,4)$, and $(6,0)$. Find the area of the convex hull that contains all four of these circles.

A. $28 + 4\sqrt{2} + \pi$ B. $28 + 4\sqrt{2} + 4\pi$
C. $36 + \pi$ D. $36 + 4\pi$ E. NOTA

23. Find the value(s) of x that satisfies the Mean Value Theorem for Integrals for the function $y = x^2 - 6x + 1$ over the interval $[1,4]$.

A. 2 B. 3 C. 4 D. 2 and 4 E. NOTA

24. Let R be the region in the first quadrant bounded by the graphs of $y = \sqrt{x} + 1$ and $x = 4$. Let the slope of the unique line passing through the origin that splits R into two subregions of equal area be equal to $\frac{A}{B}$. Find $A + B$.

A. 5 B. 14 C. 17 D. 19 E. NOTA

25. A hexagon is inscribed in a circle. Five of its sides have length 16, and the sixth, denoted by $\overline{AB}$, has length 5. If the sum of the lengths of the three diagonals that can be drawn from A is $X + Y\sqrt{Z}$ for squarefree Z, find $X + Y + Z$.
Hint: By Ptolemy's Theorem, the sum of the products of the lengths of opposite sides of a cyclic quadrilateral is equal to the product of the lengths of the diagonals.

A. 47 B. 53 C. 59 D. 61 E. NOTA

26. If the surface area of the surface generated by revolving the graph of $f(x) = \sqrt{1 - x};\ x \in \left[0, \frac{1}{2}\right]$ over the x-axis is equal to $\frac{\pi}{A}\left(B\sqrt{B} - C\sqrt{C}\right)$ for squarefree B and C, find $A + B + C$.

A. 10 B. 11 C. 13 D. 14 E. NOTA

27. The region(s) between the x-axis and graph of $y = x^2 + bx + c$ between $x = 0$ and $x = 1$ is rotated about the x-axis. Find the minimum total volume of the solid(s) formed.

A. $\frac{\pi}{360}$ B. $\frac{\pi}{180}$ C. $\frac{\pi}{80}$ D. $\frac{\pi}{30}$ E. NOTA

28. Mako has been put in timeout. She is connected by a rope around her waist to a corner of a rigid square building with a side length of 8 feet. The rope is 12 feet long. Given that the rope cannot stretch and is of negligible thickness, and that Mako is no bigger than a point, find the area outside the house that Mako can roam around in timeout in square feet.

A. 56π B. 64π C. 116π D. 124π E. NOTA

29. Complex numbers can sometimes be used to find the areas under real integrals! For example, evaluate $\int_0^\infty \frac{\cos z}{\sqrt{z}}\,dz$, where $\cos z = \Re\left(e^{iz}\right)$ and $\int \Re\left(f(z)\right) dz = \Re\left(\int f(z)\,dz\right)$.

A. $\frac{\sqrt{\pi}}{2}$ B. $\sqrt{\frac{\pi}{2}}$ C. $\sqrt{\pi}$ D. $\sqrt{2\pi}$ E. NOTA

30. Let the diameter of a circle have length r. Find the area of the circle.

A. πr B. $2\pi r$ C. πr^2 D. $2\pi r^2$ E. NOTA