CAADB CCCAC ABDCB ADBDB BAADB DBCBE

1. C The Shoelace Theorem would be long and tedious here; a geometric approach should be used instead. The figure consists of three trapezoids. Each one is an isosceles right triangle with side length 2 with an isosceles right triangle of side length 1 cut out. The total area is $3\left(\frac{2^2}{2}-\frac{1^2}{2}\right)=\frac{9}{2}$.

2. A The Shoelace Theorem should be used here instead. $\frac{1}{2}\left(\begin{vmatrix}3 & 0\\4 & 2\end{vmatrix}+\begin{vmatrix}4 & 2\\6 & 9\end{vmatrix}+\begin{vmatrix}6 & 9\\2 & 5\end{vmatrix}+\begin{vmatrix}2 & 5\\3 & 0\end{vmatrix}\right)=\frac{1}{2}(6+24+12-15)=\frac{27}{2}$.

3. A The center of Sucy's circle must lie on between $r=4$ and $r=7$, an area of $\pi(49-16)=33\pi$. The area that the center can possibly be in is between $r=1$ and $r=10$, an area of $\pi(100-1)=99\pi$. $\frac{33\pi}{99\pi}=\frac{1}{3}$.

4. D In terms of geometric probability, let Akko be the x-axis, Diana be the y-axis, and the time range be a square with opposite vertices at the origin and the point $(2,2)$. Akko will remain at the shoppe for $\frac{1}{2}$ hours, so she can be represented by the line $y=x-\frac{1}{2}$. Similarly, Diana's line is $y=x+\frac{2}{3}$. The hexagon contained in the square and these two lines represents the two being able to have their date. Its area is the square minus two isosceles right triangles, or $4-\frac{(3/2)^2}{2}-\frac{(4/3)^2}{2}=4-\frac{9}{8}-\frac{8}{9}=\frac{143}{72}$, so the probability is one-fourth this, or $\frac{143}{288}$. $143+288=431$.

5. B The volume is the double integral $\int_0^{2\pi/3}\int_0^{\pi/4}\left(1+\frac{1}{4}\sin\big(4(x+y)\big)\right)\,dy\,dx=\int_0^{2\pi/3}\left(\frac{\pi}{4}+\frac{1}{8}\cos(4x)\right)\,dx=\frac{\pi^2}{6}-\frac{\sqrt{3}}{64}$.

6. C The distance is $\frac{|3(-2)+4(5)+5(-1)-12|}{\sqrt{3^2+4^2+5^2}}=\frac{|-6+20-5-12|}{\sqrt{50}}=\frac{3}{5\sqrt{2}}=\frac{3\sqrt{2}}{10}$.

7. C A frustum is a cone minus another cone. The larger cone has base 16π and height 12, and the smaller cone has base 4π and height 6, since the radius of the smaller cone is half the radius of the larger cone. The volume of the frustum is $\frac{16\pi}{3}\cdot 12-\frac{4\pi}{3}\cdot 6=64\pi-8\pi=56\pi$.

8. C The bases have a total area of 20π. The lateral surface area of the larger cone is $4\pi\cdot 4\sqrt{10}=16\pi\sqrt{10}$. The lateral surface area of the smaller cone is $2\pi\cdot 2\sqrt{10}=4\pi\sqrt{10}$. Thus, the lateral surface area of the frustum is $20\pi+12\pi\sqrt{10}$.

9. A This radius is also equal to the radius of the incircle of a triangle with the same base and height as the cone, which are 8 and 12 respectively. The identical legs of the isosceles triangle are $4\sqrt{10}$. By Heron's formula, the radius of the incircle of an isosceles triangle with base b and legs of length a is $\frac{b}{2}\sqrt{\frac{2a-b}{2a+b}}$, which here is $\frac{4}{3}\left(\sqrt{10}-1\right)$. $4+3+10+1=18$.

10. C $b^2-4ac=8^2-4\cdot 3\cdot 3=64-36=28>0$, so this is a hyperbola.

11. A By inspection, these intersect at $x = \frac{3\pi}{4}$. The volume of the region is $\pi \int_0^{3\pi/4} \left((2 + \sin(2x))^2 - \left(\frac{4x}{3\pi}\right)^2 \right) \, dx = \pi \int_0^{3\pi/4} \left(4 + 4\sin(2x) + \frac{1-\cos(4x)}{2} - \frac{16x^2}{9\pi^2} \right) \, dx = \pi \left[-\frac{16x^3}{27\pi^2} + \frac{9x}{2} - \frac{\sin(4x)}{8} - 2\cos(2x) \right]_0^{3\pi/4} = 2 + \frac{25\pi}{8}$. $2 + 25 + 8 = 35$.

12. B The largest triangle that can fit in a circle is an equilateral triangle, whose area is $\frac{3\sqrt{3}}{4\pi}$ that of the whole circle. An ellipse is a stretched circle, and this optimization holds. The area of the ellipse is $2 \cdot 45\pi = 90\pi$, so the maximum area of the triangle is $\frac{135\pi\sqrt{3}}{2}$.

13. D The height of the box is x, and the dimensions of its base are $6 - 2x$ and $9 - 2x$ for a total volume of $4x^3 - 30x^2 + 54x$. The derivative of this is $12x^2 - 60x + 54$, which has a positive root of $\frac{5-\sqrt{7}}{2}$. This is in fact a local maximum. Plugging this into the (factored) volume formula gives a volume of $10 + 7\sqrt{7}$. $10 + 7 + 7 = 24$.

14. C This shape is the rotation of the line $y = b$ in the first quadrant over the line $y = -2$ between $x = 0$ and $x = a^3$. Its volume is $\pi \int_0^{a^3} ((b+2)^2 - 4) \, dx = \pi a^3 (b^2 + 4b)$. It is given that this equals 96π, so $a^3 = \frac{96}{b^2+4b}$. Deriving implicitly, $3a^2 \frac{da}{dt} = \frac{192(b+2)}{(b^2+4b)^2} \frac{db}{dt}$. When $a = 1$, $b = 8$, so $3\frac{da}{dt} = \frac{1920}{96^2}\frac{db}{dt}$ and $\frac{da}{dt} = \frac{5}{72}\frac{db}{dt} = \frac{5}{8}$.

15. B Four sides of the dodecagon must be on the square. Drawing the square shows a pair of $30 - 60 - 90$ triangles whose hypotenuse is 2 and a square with side length 1 between the large square and the dodecagon. The desired area is $4\left(2 \cdot \frac{\sqrt{3}}{2} + 1\right) = 4 + 4\sqrt{3}$.

16. A To the right of the y-axis, this is the area in the circle, $\frac{\pi}{2}$. To the left, this is twice the area inside $r = 1 + \cos\theta$ for $\frac{\pi}{2} \le \theta \le \pi$, which is $\int_{\pi/2}^{\pi} (1 + \cos\theta)^2 \, d\theta = \int_{\pi/2}^{\pi} \left(\frac{3}{2} + 2\cos\theta + \frac{\cos(2\theta)}{2} \right) \, d\theta = \frac{3\pi}{4} - 2$. Summing these together gives $\frac{5\pi}{4} - 2$.

17. D Since $L = \int_{-\pi}^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$, the distance that Sirius walks is equal to $5\int_{-\pi}^{\pi} \sqrt{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta} \, d\theta = 10\int_0^{\pi} \sqrt{2 + 2\cos\theta} \, d\theta = 10\int_0^{\pi} \sqrt{4\cos^2\frac{\theta}{2}} \, d\theta = 20\int_0^{\pi} \cos\frac{\theta}{2} \, d\theta = 40$.

18. B $x \ge 0$ when $t \ge 2$ and $y \ge 0$ when $0 \le t \le 3$, so our bounds are 2 and 3. $\int_2^3 y \, dx = \int_2^3 2(3t - t^2) \, dt = 3t^2 - \frac{2t^3}{3}\Big]_2^3 = 9 - \frac{20}{3} = \frac{7}{3}$.

19. D Integrating by parts can show that $\int e^{-x} \sin x \, dx = -\frac{e^{-x}}{2}(\sin x + \cos x) + C$. The integral is equal to $\lim_{n\to\infty} \sum_{k=0}^{n-1} \int_{k\pi}^{(k+1)\pi} e^{-x} \sin x \, dx = \lim_{n\to\infty} \sum_{k=0}^{n-1} \left[-\frac{e^{-x}}{2}(\sin x + \cos x) \right]_{k\pi}^{(k+1)\pi} = \lim_{n\to\infty} \frac{1}{2} \sum_{k=0}^{n-1} \left(e^{-k\pi} + e^{-(k+1)\pi} \right)$. This is the sum of two geometric

series with ratio $e^{-\pi}$, one having first term $\frac{1}{2}$ and the other having first term $\frac{e^{-\pi}}{2}$. Thus, the integral is equal to $\frac{1+e^{-\pi}}{2(1-e^{-\pi})} = \frac{e^{\pi}+1}{2(e^{\pi}-1)}$.

20. C Because the equal step size is 2, this approximation is $\left(-f(-1) + f(1)\right) + \left(f(1) + 3f(3)\right) + \left(3f(3) + 5f(5)\right) = 3 - 2 + 26 = 27$.

21. B Factor as $y''(x+y) + y'(1+y')$. The expression becomes the product rule expression for the derivative of $(x+y)y'$ with respect to x. Evaluated at 6 and 1, the integral equals $54 - 32 = 22$.

22. A The area of the trapezoid formed by the points is 16. Because of the convex hull, rectangles of height 1 need to be drawn on the perimeter of the trapezoid, which is $12 + 4\sqrt{2}$. Finally, adding the area of the parts of the circles gives a total convex hull area of $28 + 4\sqrt{2} + \pi$.

23. A $\frac{1}{3}\int_1^4 (x^2 - 6x + 1)\,dx = \left[\frac{x^3}{9} - x^2 + \frac{x}{3}\right]_1^4 = -7$. Solving $x^2 - 6x + 1 = -7$ gives $x = 2$ or $x = 4$; however, the value that satisfies the Mean Value Theorem for Integrals must lie in the open interval, so 4 must be excluded and 2 is the only solution.

24. D $\int_0^4 \left(\sqrt{x} + 1\right) dx = \left[\frac{2x^{3/2}}{3} + x\right]_0^4 = \frac{28}{3}$. The line forms a triangle with base 4 and area $\frac{14}{3}$, so its height must be $\frac{7}{3}$. The slope of the line is $\frac{7}{12}$. $7 + 12 = 19$.

25. B Let $x = AC = BF$, $y = AD = BE$, and $z = AE = BD$. By Ptolemy's on $ABCD$, $16y + 80 = xz$, and by Ptolemy's on $ACDF$, $xz + 256 = y^2$. Subtracting these gives $y^2 - 16y - 336 = 0$. This gives the root $y = 28$. Ptolemy's on $ADEF$ gives $16y + 256 = z^2$, so $z = 8\sqrt{11}$. Plugging this into the first equation gives $x = 6\sqrt{11}$. The sum of the lengths of the diagonals is $28 + 14\sqrt{11}$. $28 + 14 + 11 = 53$.

26. D $f'(x) = -\frac{1}{2\sqrt{1-x}}$, so $2\pi\int_0^{1/2} \sqrt{1-x}\sqrt{1 + \frac{1}{4(1-x)}}\,dx = 2\pi\int_0^{1/2} \frac{\sqrt{5-4x}}{2}\,dx = \frac{\pi}{4}\int_3^5 \sqrt{u}\,du = \left[\frac{\pi}{6}u^{3/2}\right]_3^5 = \frac{\pi}{6}\left(5\sqrt{5} - 3\sqrt{3}\right)$. $6 + 5 + 3 = 14$.

27. B The volume is $\pi\int_0^1 (x^2 + bx + c)^2\,dx = \pi\int_0^1 (x^4 + 2bx^3 + (b^2 + 2c)x^2 + 2bcx + c^2)\,dx = \pi\left(\frac{1}{5} + \frac{b}{2} + \frac{b^2 + 2c}{3} + bc + c^2\right) = \pi K$. $\frac{\partial K}{\partial b} = \frac{2b}{3} + c + \frac{1}{2}$ and $\frac{\partial K}{\partial c} = b + 2c + \frac{2}{3}$. Setting these expressions equal to 0 and solving the system of equations yields $b = -1$ and $c = \frac{1}{6}$. This is the absolute minimum, and evaluating the expression with these values yields a minimum volume of $\frac{\pi}{180}$.

28. C The point of attachment of the rope can see $\frac{3}{4}$ of a circle with radius 12, with an area of 108π. The rope can also wrap around the house with a radius of 4, forming two quarter circles. The area of these is $2 \cdot \frac{1}{4} \cdot 16\pi = 8\pi$, for a total area of 116π.

29. B Noting the z that usually denotes complex values, this integral is equal to $\Re\int_0^\infty \frac{e^{iz}}{\sqrt{z}}\,dz = \Re\int_0^\infty 2e^{iu^2}\,du = \Re\int_{-\infty}^\infty e^{iu^2}\,du$. Noting that $\int_{-\infty}^\infty e^{au^2}\,du = \sqrt{\frac{\pi}{-a}}$, the complex integral is equal to $\sqrt{\frac{\pi}{-i}} = \sqrt{\pi}\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$. The real part of this is $\sqrt{\frac{\pi}{2}}$.

30. E Consider a disk with radius a. Then the area of the disk is the double integral of 1 over the disk itself, D. This can then be converted to a double integral and solved. $\iint_D 1\,d(x,y) = \iint_D t\,dt\,d\theta = \int_0^a\int_0^{2\pi} t\,d\theta\,dt = \int_0^a 2\pi t\,dt = \pi a^2$. Setting $a = \frac{r}{2}$ gives the area $\frac{\pi r^2}{4}$.