Mu Ciphering Nationals 2023 solutions

0. $y' = \cos x + 1 \to y' = 2 \to y = 2x$

1. Start this as a stars and bars question. Subtract out the 12 scenarios where 2 of the numbers are the same. Those can each be arranged 3 ways. Divide by 6 because once you have 3 different numbers there is only 1/6 that yield the correct order. $\frac{{}_{24}C_2 - 3(12)}{{}_{24}C_2} = \frac{240}{276} \to \frac{240}{276} \bullet \frac{1}{6} = \frac{10}{69}$

$$\int_1^2 \frac{9x+4}{x^5+3x^2+x}dx = \ln\frac{L}{U} \to u = x^5+3x^2+x \to du = (5x^4+6x+1)dx$$

2. $9x+4 = \left(5x^4+15x+5\right)-\left(5x^4+6x+1\right) \to \int_1^2 \frac{\left(5x^4+15x+5\right)-\left(5x^4+6x+1\right)}{x^5+3x^2+x}dx$

$$\int_1^2 \frac{5}{x}dx - \int_5^{46}\frac{1}{u}du = 5\ln 2 - \ln 46 + \ln 5 = \ln\frac{80}{23} \to 103$$

3. Draw yourself a picture:

$$b+h=31 \to b=31-h \to 2x+b=50 \to x=25-\frac{b}{2} \to h^2 = x^2-\left(\frac{b}{2}\right)^2$$

$$h^2 = \left(25-\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 = 625-25b = 625-25(31-h)$$

$$h^2-25h+150=0 \to (h-10)(h-15)=0 \to 25$$

4. $C = 2\left(x^2+4xh\right)+5x^2 = 7x^2+8xh = 7x^2+\frac{1792}{x}$

$$C' = 14x - \frac{1792}{x^2} = 0 \to x = 4\sqrt[3]{2} \to h = \frac{224}{16\sqrt[3]{4}} = 7\sqrt[3]{2}$$

5. $\frac{k+\sqrt{k}+4-2k+2}{k-1} < 0 \to \frac{-k+\sqrt{k}+6}{k-1} < 0 \to \frac{k-\sqrt{k}-6}{1-k} < 0$

$$\frac{\left(\sqrt{k}-3\right)\left(\sqrt{k}+2\right)}{1-k} < 0 \to [0,1)\cup(9,\infty) \to 10-2023, 0 \to 2014+1 = 2015$$

6. $2\pi\int_0^1 (2-x)\left(2x-5x^{\frac{2}{3}}+3\right)dx \to 2\pi\int_0^1\left(-2x^2+5x^{\frac{5}{3}}+x-10x^{\frac{2}{3}}+6\right)dx$

$$2\pi\left(\frac{-2}{3}x^3+\frac{15}{8}x^{\frac{8}{3}}+\frac{1}{2}x^2-6x^{\frac{5}{3}}+6x\right)\Big|_0^1 = 2\pi\left(\frac{-2}{3}+\frac{15}{8}+\frac{1}{2}-6+6\right) = \frac{41\pi}{12}$$

7. Draw yourself a good picture: Since WF is a diameter angle WZF is a right angle, and the triangle is a 10-24-26 right triangle. You can then use angle bisector theorem: You can call LU=x
$\frac{10}{13-x}=\frac{24}{13+x}\rightarrow 130+10x=312-24x\rightarrow 34x=182\rightarrow x=\frac{91}{17}\rightarrow 108$
8. $\frac{1}{2}\left(\frac{f}{\sqrt{2}}\right)\left(\frac{f}{\sqrt{2}}\right)=\frac{1}{4}f^2 \cdot \frac{1}{4}\int_0^2 (2x-x^2)^2 dx$. $\frac{1}{4}\left(\frac{4}{3}x^3-\frac{4x^4}{3}+\frac{1}{5}x^5\right)$
evaluate from 0 to 2= $\frac{4}{15}$
9. The sum is 78. He has a sum of 54 already and therefore his remaining 3 scores must sum to 24. The median will be the average of the 5^{th} and 6^{th} number. If you play around a little the only 3 possible values are 7, 7.5 and 8. Therefore the sum is 22.5
10. The intersections occur at $\theta=\frac{\pi}{2},\frac{3\pi}{2}$ .You can exploit some symmetry and break into 4 pieces fo equal area:

$$4\bullet\frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} r^2 d\theta = 2\int_{\frac{\pi}{2}}^{\pi}\left(2+2\cos\theta\right)^2 d\theta = 8\int_{\frac{\pi}{2}}^{\pi}\left(1+2\cos\theta+\cos^2\theta\right)d\theta$$

$$8\int_{\frac{\pi}{2}}^{\pi}\left(\frac{3}{2}+2\cos\theta+\frac{1}{2}\cos 2\theta\right)d\theta = 8\left(\frac{3\theta}{2}+2\sin\theta+\frac{1}{4}\sin 2\theta\right)\Bigg|_{\frac{\pi}{2}}^{\pi} = 6\pi-16\rightarrow 22$$

11. You need to do some case work here. Start with arithmetic with common difference of 1. We have 6 that are in decreasing and 7 that are decreasing for a total of 13. If the common difference is 2, we have 3 increasing and 4 decreasing for a total of 7. Only 1 case with common difference of 3 and that is a decreasing sequence (9,6,3,0). 13+7+1=21

12. $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{n^2+k^2}=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\cdot\frac{1}{1+\left(\frac{k}{n}\right)^2}=\int_0^1\frac{1}{1+x^2}dx=\arctan(x)\ \Big|_0^1=\frac{\pi}{4}$

Answers:

0. Y=2x
1. $\frac{10}{69}$
2. 103
3. 25
4. $7\sqrt[3]{2}$
5. 2015
6. $\frac{41\pi}{12}$
7. 108
8. $\frac{4}{15}$
9. 22.5
10. 22
11. 21
12. $\frac{\pi}{4}$