The answer choice E. NOTA denotes that 'none of these answers' are correct. All coins and dice are fair unless otherwise stated, and all random choices are uniformly random and independent unless otherwise stated. The notation $\mathbb{E}[X]$ denotes the expected value of a random variable X. Good luck and have fun!

1. Legosi randomly draws a card from a standard deck. What is the probability his card is a 7 or a club (or both)?

A. $\frac{4}{13}$ B. $\frac{15}{52}$ C. $\frac{17}{52}$ D. $\frac{1}{52}$ E. NOTA

2. A real number x is chosen at random from the interval $[0,10]$. Find the probability that $x^2 + x > 12$.

A. $\frac{2}{5}$ B. $\frac{3}{10}$ C. $\frac{3}{5}$ D. $\frac{7}{10}$ E. NOTA

3. For an appropriate constant A, the function $f: [0,1] \to \mathbb{R}$ given by $f(x) = A\sqrt{x}$ is a probability density function modeling a continuous random variable taking values from 0 to 1 inclusive. Find A.

A. 2 B. $\frac{3}{2}$ C. $\frac{2}{3}$ D. $\frac{1}{2}$ E. NOTA

4. Two fair six-sided dice are rolled. Find the probability that the sum of the rolls is at least 4.

A. $\frac{5}{6}$ B. $\frac{1}{12}$ C. $\frac{11}{12}$ D. $\frac{1}{6}$ E. NOTA

5. A fair coin is flipped five times. Find the probability that exactly three of the flips land heads.

A. $\frac{5}{32}$ B. $\frac{5}{16}$ C. $\frac{3}{16}$ D. $\frac{3}{32}$ E. NOTA

6. How many distinct arrangements are there of the letters in the word BEASTARS?

A. 5040 B. 10080 C. 40320 D. 20160 E. NOTA

7. What is the probability that two randomly selected cards from a standard deck are of the same rank?

A. $\frac{1}{17}$ B. $\frac{1}{13}$ C. $\frac{1}{4}$ D. $\frac{4}{17}$ E. NOTA

8. A real number a is chosen at random from the interval $\left[0, \frac{3\pi}{2}\right]$. Find the probability that $\int_0^a \sin(x)\,\mathrm{d}x > \frac{1}{2}$.

A. $\frac{2}{9}$ B. $\frac{5}{9}$ C. $\frac{4}{9}$ D. $\frac{7}{9}$ E. NOTA

9. There are six dogs and five cats in the drama club (all animals are distinguishable). A joint committee of cats and dogs with five animals is to be chosen. If the absolute difference between the number of cats on the committee and the number of dogs on the committee is at most 1, find the total number of possible distinct committees.

A. 350 B. 400 C. 150 D. 300 E. NOTA

10. How many polynomials P with non-negative integer coefficients satisfy $P(0) = 0$ and $P'(1) = 4$?

A. 6 B. 4 C. 5 D. 7 E. NOTA

11. Collot has ten cards, numbered 1-10. He picks five cards at random and places them face down in ascending order. Given that he then turns over the third card to reveal a 4, find the probability that the first card is a 1.

A. $\frac{2}{3}$ B. $\frac{1}{2}$ C. $\frac{3}{4}$ D. $\frac{3}{5}$ E. NOTA

12. Tao is standing on the real number line. He starts at 0, and each second, he randomly takes one one-unit step in either the $+1$ direction or the -1 direction, with either option equally likely. Find the probability that in his first 8 steps, Tao never reaches -2 or 2.

A. $\frac{81}{256}$ B. $\frac{91}{128}$ C. $\frac{1}{128}$ D. $\frac{1}{16}$ E. NOTA

13. Bill repeatedly rolls a fair 20-sided die until he gets the number 20. What is the probability it takes him an odd number of rolls to do so?

A. $\frac{9}{20}$ B. $\frac{20}{39}$ C. $\frac{19}{39}$ D. $\frac{11}{20}$ E. NOTA

14. A complex number z satisfying $|z| = 1$ is selected uniformly at random. Find the probability that $|z + 1| > 1$.

A. $\frac{1}{2}$ B. $\frac{2}{3}$ C. $\frac{3}{4}$ D. $\frac{5}{6}$ E. NOTA

15. On any given flip, a certain magical coin has a 48% chance of landing heads, a 48% chance of landing tails, and a 4% chance of disappearing into thin air, never to be seen again. If Voss repeatedly flips the coin until it disappears, what is the expected number of times it will land heads?

A. 12 B. 12.5 C. 24 D. 25 E. NOTA

16. A point is selected uniformly at random inside square $ABCD$ of side length $\sqrt{2}$. Find the expected distance from this point to diagonal $\overline{AC}$.

A. $\frac{1}{4}$ B. $\frac{1}{2}$ C. $\frac{1}{6}$ D. $\frac{1}{3}$ E. NOTA

17. A full house is a poker hand consisting of three cards of the one rank along with a pair of cards from a different rank. A standard deck of cards is randomly shuffled, and Haru is dealt three cards: the 7 of spades, the Jack of hearts, and the 7 of clubs. If she is then dealt two more cards, what is the probability she gets a full house?

A. $\frac{1}{192}$ B. $\frac{1}{221}$ C. $\frac{3}{392}$ D. $\frac{3}{442}$ E. NOTA

18. Let X be a continuous random variable over $\mathbb{R}$ modeled by the probability density function $f(x)$, and define the 3-*variance* of X to be $\text{Var}_3(X) = \int_{-\infty}^{\infty} (x - \mathbb{E}[X])^3 f(x)\,\mathrm{d}x$. If $\mathbb{E}[X] = 1$, $\mathbb{E}[X^2] = 4$, and $\mathbb{E}[X^3] = 16$, compute $\text{Var}_3(X)$.

A. 15 B. 4 C. 6 D. 7 E. NOTA

19. Pina is standing in the Argand plane. How many possible paths could he take from 0 to 4 that do not pass through 2 and consist only of one-unit steps in the $+1$, $+e^{i(\pi/3)}$, and $+e^{i(-\pi/3)}$ directions?

A. 212 B. 321 C. 169 D. 152 E. NOTA

20. An unfair coin has probability p of landing heads on any given flip. If the coin is flipped three times and you are told that at least two of the flips landed heads, the probability that all three flips landed heads is $\frac{1}{5}$. If p can be written in simplest form as $\frac{m}{n}$, compute $m + n$.

A. 12 B. 11 C. 9 D. 10 E. NOTA

21. Find the largest positive integer k such that the following holds:

$$\binom{5}{0}\binom{6}{0}+\binom{5}{1}\binom{6}{1}+\binom{5}{2}\binom{6}{2}+\cdots+\binom{5}{5}\binom{6}{5}=\binom{11}{k}$$

A. 4 B. 5 C. 6 D. 7 E. NOTA

22. Jack generates a subsequence of the harmonic sequence $\frac{1}{1},\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5},\ldots$ as follows: Jack includes the number $\frac{1}{n}$ in his subsequence with probability $\frac{1}{2^n}$. Compute the expected sum of Jack's subsequence.

A. $\frac{\sqrt{2}}{2}$ B. $\frac{2}{3}$ C. $\ln(2)$ D. $\frac{\pi}{4}$ E. NOTA

23. Real numbers a and b are chosen at random from the interval $[0,10]$. Find the probability that the area of the finite region bounded above by the graph of $y=-x^2+ax+b$ and below by the x-axis is at least 36.

Hint: The area bounded by $-(x-r)(x-s)=-x^2+ax+b$ is dependent on $|r-s|$ only.

A. $\frac{12}{25}$ B. $\frac{16}{25}$ C. $\frac{13}{25}$ D. $\frac{9}{25}$ E. NOTA

24. Louis flips a coin repeatedly, stopping once he gets his fourth head. What is the probability that he flips the coin an even number of times?

A. $\frac{13}{27}$ B. $\frac{14}{27}$ C. $\frac{40}{81}$ D. $\frac{41}{81}$ E. NOTA

25. For an appropriate constant A, The function $f:[0,\infty)\to\mathbb{R}$ given by $\frac{A}{(1+x^4)(1+x^8)}$ is a probability density function modeling a continuous random variable X taking non-negative real values. Compute $\mathbb{E}[X]$ in terms of A.

A. $\frac{A\pi}{4}$ B. $\frac{A\pi}{8}$ C. $\frac{A\pi}{2}$ D. $A\pi$ E. NOTA

26. Free and Agata are playing a game with a fair coin. Starting with Free, they take turns flipping the coin, recording the sequence of outcomes. The first time they obtain the sequence HHT (H = heads, T = tails), they stop flipping the coin, and whoever flipped the last T wins. Find the probability that Free wins the game.

A. $\frac{6}{11}$ B. $\frac{8}{15}$ C. $\frac{7}{15}$ D. $\frac{5}{11}$ E. NOTA

27. Regular hexagon $ABCDEF$ has side length $4\sqrt{3}$, and diagonals $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are marked. A circular coin of radius 1 is dropped randomly such that it lies entirely within $ABCDEF$. Find the expected number of marked diagonals crossed by this coin.

A. $\frac{14}{25}$ B. $\frac{16}{25}$ C. $\frac{18}{25}$ D. $\frac{19}{25}$ E. NOTA

28. A point is selected uniformly at random on the interior of the region bounded by the polar curve given by the equation $r = 1 - \theta^2$ for $-1 \le \theta \le 1$. Find the expected distance between this point and the pole (the origin).

A. $\frac{4}{7}$ B. $\frac{3}{5}$ C. $\frac{8}{15}$ D. $\frac{16}{105}$ E. NOTA

29. Durham and Miguno start at $(0,0)$ in the coordinate plane. Every second, they each randomly step one unit in the $+x$ direction or one unit in the $+y$ direction. Let $P(n)$ denote the probability that after each of them takes n steps they end up at the same point. There exists a unique real r such that

$$K = \lim_{n\to\infty} n^r \, P(n)$$

exists and is nonzero. Compute r.

A. 1 B. $\frac{1}{2}$ C. $\frac{3}{2}$ D. 2 E. NOTA

30. In the framework of the previous problem, compute K.

A. $\frac{1}{\sqrt{\pi}}$ B. $\sqrt{\pi}$ C. $\frac{1}{\sqrt{2\pi}}$ D. $\sqrt{2\pi}$ E. NOTA