For all questions, NOTA means "None of the Above." A standard deck of 52 cards has 4 suits (clubs, hearts, diamonds, and spades) which each have 13 cards ranging from ace to king. A fair six-sided die has faces numbered one through six. Good luck and have fun!

1. The number of permutations of the letters of the word JERMYGRIFF can be expressed in the form $\frac{n!}{m^2}$ where n and m are positive integers and n is minimized. What is $n+m$?

   A. 9 B. 10 C. 11 D. 12 E. NOTA

2. How many 6-digit positive integers have the property that their digits are strictly decreasing from left to right (e.g. 976541 but not 877652)?

   A. 84 B. 126 C. 210 D. 252 E. NOTA

3. 4 people are playing a card game. A standard deck of 52 cards is completely dealt to the 4 players so that each person has 13 cards. What is the probability that one person is dealt all 13 spades?

   A. $\frac{1}{\binom{52}{13}}$ B. $\frac{2}{\binom{52}{13}}$ C. $\frac{4}{\binom{52}{13}}$ D. $\frac{24}{\binom{52}{13}}$ E. NOTA

4. Two real numbers a and b are independently chosen uniformly at random from the interval $[-6,6]$. What is the probability that the magnitude of $a+bi$ is less than $4\sqrt{3}$?

   A. $\frac{\pi}{3}$ B. $\frac{\pi}{12}$ C. $\frac{\pi+3\sqrt{3}}{9}$ D. $\frac{\pi+3\sqrt{3}}{36}$ E. NOTA

5. In how many distinguishable ways can 8 people, Jeremy and Julian among them, stand in a circle such that Jeremy and Julian are not standing next to each other? Two arrangements are considered the same if one can be rotated to obtain the other.

   A. 1440 B. 3600 C. 4320 D. 38880 E. NOTA

6. The probabilities of events A and B are 0.7 and 0.9 respectively. The largest possible probability of A and B both occurring is w and the smallest possible probability is x. The largest possible probability of either A or B occurring is y and the smallest possible probability is z. What is the value of $w + x + y + z$?

A. 2 B. 2.3 C. 2.6 D. 3.2 E. NOTA

7. A value of a is chosen uniformly at random from the interval $[1, 3]$. What is the expected value of the slope of the line normal to $y = \ln x$ at $x = a$?

A. -2 B. 2 C. $\frac{\ln 3}{2}$ D. $\frac{-\ln 3}{2}$ E. NOTA

8. A random point inside the region $x^2 + y^2 + z^2 \leq 1$ is chosen with all points being equally likely. What is the probability that the point has a y-coordinate greater than $\frac{1}{2}$?

A. $\frac{5}{128}$ B. $\frac{5}{64}$ C. $\frac{5}{32}$ D. $\frac{5}{16}$ E. NOTA

9. A point is chosen in the region bounded by $r = 4\cos\left(\theta - \frac{\pi}{4}\right)$. What is the probability that the point is also inside the region bounded by $r^2 = \sin(2\theta)$?

A. $\frac{1}{32\pi}$ B. $\frac{1}{16\pi}$ C. $\frac{1}{8\pi}$ D. $\frac{1}{4\pi}$ E. NOTA

10. Julian keeps flipping a fair coin until he flips tails for the first time. What is the expected number of times he flips the coin?

A. 1.5 B. 2 C. 2.5 D. 4 E. NOTA

11. If n is a randomly selected natural number between 1 and 216 inclusive, what is the probability that $\frac{n}{72}$ is in simplest form?

A. $\frac{1}{4}$ B. $\frac{1}{3}$ C. $\frac{1}{2}$ D. $\frac{2}{3}$ E. NOTA

For questions 12 and 13, refer to the following information.

I have some identical, unfair 11-sided dice with faces numbered from 1 to 11. The probabilities of rolling a 1, 2, 3, 4, 5, or 6 are directly proportional to the number (i.e. a 2 is twice as likely to be rolled as a 1, a 6 is three times as likely to be rolled as a 2, etc.). If two numbers sum to 12, they are equally likely to be rolled. 31 of these dice are rolled at the same time.

12. What is the probability that the product of the numbers shown on the faces is a multiple of 3?

A. $1-\left(\frac{1}{3}\right)^{31}$ B. $\left(\frac{1}{3}\right)^{31}$ C. $1-\left(\frac{2}{3}\right)^{31}$ D. $\left(\frac{2}{3}\right)^{31}$ E. NOTA

13. What is the probability that the sum of the numbers shown on the faces is a multiple of 3?

A. $\frac{3}{11}$ B. $\frac{4}{11}$ C. $\frac{1}{6}$ D. $\frac{1}{3}$ E. NOTA

14. A magic six-sided die has faces with 1, 2, 3, 4, 5, and D. When D is rolled, the numerical values on the dice all double. Thus, if D is rolled one time, the die now has faces with 2, 4, 6, 8, 10, and D, and if D is rolled again, the die has faces with 4, 8, 12, 16, 20, and D. I keep rolling the die until I get a numerical value (i.e. until I don't roll a D). If each face has an equal chance of being rolled, is the expected value of the number I roll?

A. 3 B. 3.25 C. 3.5 D. 3.75 E. NOTA

15. A mail carrier has 10000 letters for 10000 different people. He decides to hand them out randomly without regard to whose letter is whose. Which of the following is closest to the probability that no one receives the correct letter?

A. $\frac{1}{e}$ B. $\frac{1}{2}$ C. $\frac{2}{3}$ D. $\frac{3}{4}$ E. 1

16. The probability p of an event occurring is a solution to the equation $15p^3+23p^2-4p-12=0$. What is the sum of the numerator and denominator of p when it is written in simplest fractional form?

A. 3 B. 4 C. 7 D. 11 E. NOTA

17. I have a red button and a blue button which are both supposed to turn on a light when pressed. The red button turns on the light 100% of the time while the blue button turns it on n% of the time. I select a button at random with both buttons having equal probability of being selected. When I press the button once, the light turns on. Given the result of this first button press, there is a 37.5% chance that I selected the blue button. When the same button is pressed for the second time the light is turned on once again. Given the results of both button presses, what is the probability that the button I selected is blue?

A. $\frac{9}{34}$ B. $\frac{1}{4}$ C. $\frac{3}{8}$ D. $\frac{9}{64}$ E. NOTA

For questions 18 and 19, refer to the following information.

X is a random variable taking on only nonnegative integral values. The probability that $X = k$ is given by $\frac{n \cdot 2^k}{k!}$, where n is a constant.

18. What is the value of n such that X has a valid probability distribution?

A. $\frac{1}{e}$ B. $\frac{1}{e^2}$ C. e D. e^2 E. NOTA

19. Let Y be a random variable that is independent of X but has the same distribution as X. What is the probability (in terms of n) that $X + Y > 2$?

A. $1 - 4n^2$ B. $1 - 5n^2$ C. $1 - 9n^2$ D. $1 - 13n^2$ E. NOTA

20. Region R is bounded by the graphs of $y = x^2$ and the line $y = k$, where k is chosen uniformly at random from the interval $[1, 5]$. Region R is then rotated about line $y = k$ to form a solid. The expected value of the volume of this solid can be expressed in the form $\frac{a\sqrt{b}-c}{d}\pi$, where a, b, c, d are positive integers such that $\gcd(a, c, d) = 1$ and b is square-free. What is $a + b + c + d$?

A. 1118 B. 4142 C. 2126 D. 614 E. NOTA

21. What is the constant term in the expansion of $(x^3 - \frac{1}{x^2})^{15}$?

A. 2520 B. -2520 C. 5005 D. -5005 E. NOTA

For questions 22 and 23, refer to the following information

Jeremy has some tiles. He has 1×1 tiles that come in 2 different colors and 1×2 tiles that come in 8 different colors. (All 10 of the colors are distinct.) He is exploring how many ways there are to cover different areas with these tiles. Assume that the areas he is covering cannot be rotated or reflected, and he has sufficient number of tiles to complete any tiling using any of the 10 colors.

22. In how many ways can he cover a 1×6 area with the tiles?

A. 672 B. 2752 C. 3024 D. 8880 E. NOTA

23. Let the number of ways to cover a $1 \times n$ area be $f(n)$. What is $\lim_{n\to\infty} \frac{f(n)}{4^n}$?

A. $\frac{1}{3}$ B. $\frac{2}{3}$ C. 1 D. $\frac{4}{3}$ E. NOTA

For questions 24 and 25, refer to the following information

You are given 4 sets, *A*, *B*, *C*, and *D*. Define a_n to be the sum of the sizes of all possible sets formed by taking the intersection of sets *A*, *B*, *C*, and *D* n at a time (i.e. $a_2 = |A \cap B| + |A \cap C| + |A \cap D| + |B \cap C| + |B \cap D| + |C \cap D|$). $a_2 = 990$, $a_3 = 180, a_4 = 10$, and $|A \cup B \cup C \cup D| = 1200$.

24. What is the value of a_1?

A. 2020 B. 2030 C. 2040 D. 2050 E. NOTA

25. How many elements belong to at least two of sets *A*, *B*, *C*, and *D*?

A. 660 B. 670 C. 680 D. 690 E. NOTA

26. An arrangement of the letters of the word POSSESSIONS is considered proper if there is at least one S to both the left and right of the P. Let n be the total number of arrangements of POSSESSIONS. How many proper arrangements of the word are there in terms of n?

A. $\frac{n}{2}$ B. $\frac{2n}{3}$ C. $\frac{3n}{5}$ D. $\frac{5n}{6}$ E. NOTA

For questions 27 through 29, refer to the following information.

The continuous random variable X is defined on the interval $[0, \infty)$ and has probability density function e^{-x}.

27. What is the median of X?

A. $\frac{1}{2}$ B. 1 C. $\ln 2$ D. $\frac{1}{e^2}$ E. NOTA

28. The variance of X can be expressed as $\int_0^\infty f(x)dx - \left(\int_0^\infty g(x)dx\right)^2$ for some functions $f(x)$ and $g(x)$. Which of the following functions could be $f(x)$ and $g(x)$? For each answer choice, the first function is $f(x)$ and the second is $g(x)$.

A. xe^{-x}, x^2e^{-x} B. x^2e^{-x}, xe^{-x} C. $\frac{e^{-x}}{x^2}, \frac{e^{-x}}{x}$ D. $\frac{e^{-x}}{x}, \frac{e^{-x}}{x^2}$ E. NOTA

29. How many of the following are true?

I. This distribution is an exponential distribution
II. This distribution is the continuous analog to a geometric distribution
III. This distribution is memoryless
IV. The variance of this distribution is equal to the square of its mean

A. 0 B. 1 C. 2 D. 3 E. NOTA

30. Probabilities must be strictly greater than which of the following? (The answer is not E)

A. -1 B. 0 C. 1 D. 100 E. NOTA