1. **A** There are 52 total possible cards to choose, 13 of which are clubs, and 3 of which are of 7s that are not clubs. This gives a probability of $\frac{16}{52} = \boxed{\frac{4}{13}}$.

2. **D** Note that $x^2 + x$ is increasing over $[0,10]$ and $x^2 + x = 12$ at $x = 3$. This means that $x^2 + x > 12$ for all $x \in (3,10]$, giving a probability of $\boxed{\frac{7}{10}}$.

3. **B** In order to be a probability density function, f must satisfy $\int_0^1 f(x)\,\mathrm{d}x = 1$. This means $1 = \int_0^1 A\sqrt{x}\,\mathrm{d}x = \left[\frac{2A}{3}x^{3/2}\right]_0^1 = \frac{2A}{3}$, so $A = \boxed{\frac{3}{2}}$.

4. **C** There are $6 \cdot 6 = 36$ possible outcomes for two dice rolls. Of these, one sums to 2 and two sum to 3. This means that $36 - 1 - 2 = 33$ sum to at least 4, giving a probability of $\frac{33}{36} = \boxed{\frac{11}{12}}$.

5. **B** There are $\binom{5}{3} = 10$ ways to choose 3 of the 5 flips to land heads, and $2^5 = 32$ total possible sequences of flips, for a probability of $\frac{10}{32} = \boxed{\frac{5}{16}}$.

6. **B** If we initially pretend the A's and S's are distinct (i.e. something like BEASTaRs), there are $8! = 40320$ ways to arrange the letters. However, this overcounts the two ways to arrange the A's and the two ways to arrange the S's, so we need to divide by $2 \cdot 2$ to get $\boxed{10080}$.

7. **A** Regardless of the first card selected, there are 3 other cards in the deck of the same rank out of 51 total choices, for a probability of $\frac{3}{51} = \boxed{\frac{1}{17}}$.

8. **D** We integrate $\int_0^a \sin(x)\,dx = [-\cos(x)]_0^a = 1 - \cos(a)$. We want this to be greater than $\frac{1}{2}$, which occurs when $\cos(a) < \frac{1}{2}$. This occurs when $a \in \left[\frac{\pi}{3}, \frac{3\pi}{2}\right]$, which gives a probability of $\frac{3\pi/2 - \pi/3}{3\pi/2} = \boxed{\frac{7}{9}}$.

9. **A** We can either pick three dogs and two cats, which gives $\binom{6}{3}\binom{5}{2} = 20 \cdot 10 = 200$ committees, or two dogs and three cats, which gives $\binom{6}{2}\binom{5}{3} = 15 \cdot 10 = 150$ committees. This gives a total of $\boxed{350}$ possible committees.

10. **C** Let the polynomial take the form $a_0 + a_1x + \cdots + a_nx^n$. $f(0) = 0$ gives that $a_0 = 0$ while $f'(1) = 4$ gives that $a_1 + 2a_2 + \cdots + na_n = 4$. Since f has non-negative integer coefficients, we know $n \le 4$, so we want $a_1 + 2a_2 + 3a_3 + 4a_4 = 4$. The numbers are small enough that we can list all possible tuples (a_1, a_2, a_3, a_4) to get

- (0,0,0,1)
- (1,0,1,0)
- (0,2,0,0)
- (2,1,0,0)
- (4,0,0,0)

for a final answer of $\boxed{5}$.

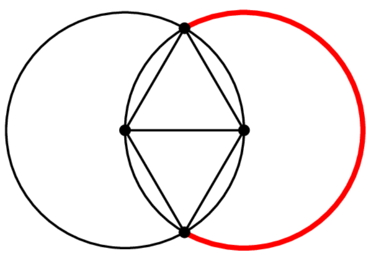
11. **A** If the third card is a 4, the first two cards are either 1 and 2, 1 and 3, or 2 and 3. There is no other information to make these not equally likely, so the probability the first card is 1 is simply $\boxed{\frac{2}{3}}$.

12. **D** Note that after his first step, Tao's distance from 0 is 1. To not hit -2 or 2, he must step back to 0 for his second step, which occurs with probability $\frac{1}{2}$. His third and fourth steps are exactly the same, as are the following two sets of two steps, so the desired probability is $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \boxed{\frac{1}{16}}$.

13. **B** The probability of Bill getting his first 20 on his $(2k+1)$th roll is $\left(\frac{19}{20}\right)^{2k}\left(\frac{1}{20}\right)$. Summing over all k, we get a probability of

$$\sum_{k=0}^{\infty}\left(\frac{19}{20}\right)^{2k}\left(\frac{1}{20}\right) = \frac{1/20}{1-(19/20)^2} = \boxed{\frac{20}{39}}.$$

14. **B**



The set of all points with $|z| = 1$ is a circle of radius 1 (the right circle in the diagram). The quantity $|z - 1|$ is the distance from z to the point $(-1,0)$ in the Argand plane, so $|z - 1| > 1$ is the set of points outside the circle centered at $(-1,0)$ of radius 1 (left circle). Notice we can draw two equilateral triangles as shown above, so the arc of points inside the circle has measure $120°$. This means that if a point is chosen at random on the right circle, there is a $\boxed{\frac{2}{3}}$ probability it lies outside the left circle.

15. **A** Let E denote the expected number of heads before the coin disappears. On any given flip, there is a 96% of still having the coin, giving another E to our expected value, with a 48% of increasing the total number of heads by 1 on this specific flip. This gives the equation $E = 0.96E + 0.48(1) \rightarrow E = \boxed{12}$.

16. D Rotate the square such that opposite vertices are at $(-1,0)$ and $(1,0)$. By symmetry, the expected distance is equivalent to finding the y-coordinate of the centroid of $y = 1 - |x|$ over $x \in (0,1)$. This is given by

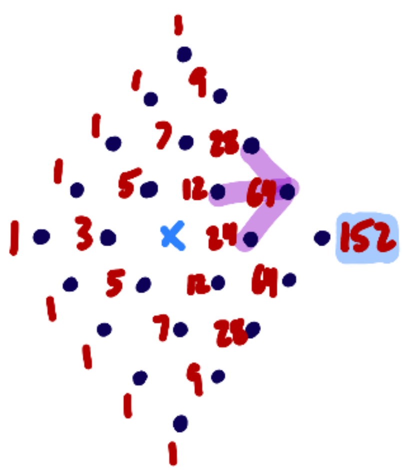
$$\frac{\int_0^1 \frac{1}{2}(1-x)^2\,\mathrm{d}x}{\int_0^1 (1-x)\,\mathrm{d}x} = \frac{\left[\frac{1}{2}x - \frac{1}{2}x^2 + \frac{1}{6}x^3\right]_0^1}{\left[x - \frac{1}{2}x^2\right]_0^1} = \frac{\frac{1}{6}}{\frac{1}{2}} = \boxed{\frac{1}{3}}.$$

17. C There are two possibilities that yield a full house: the remaining two cards are both jacks or one card is a jack and the other is a 7. The former case has $\binom{3}{2} = 3$ options, while the latter has $\binom{3}{1}\binom{2}{1} = 6$ options. There are $\binom{49}{2} = 1172$ options for the two chosen cards, so the probability is $\frac{9}{1176} = \boxed{\frac{3}{392}}$.

18. C Note that $(x - \mathbb{E}[x])^3 = x^3 - 3x^2\mathbb{E}[x] + 3x(\mathbb{E}[x])^2 - (\mathbb{E}[x])^3$, so

$$\begin{aligned}&\int_{-\infty}^{\infty} f(x)\,(x^3 - 3x^2\mathbb{E}[x] + 3x(\mathbb{E}[x])^2 - (\mathbb{E}[x])^3)\,\mathrm{d}x \\ &= \mathbb{E}[x^3] - 3\mathbb{E}[x^2]\mathbb{E}[x] + 3\mathbb{E}[x](\mathbb{E}[x])^2 - (\mathbb{E}[x])^3 \\ &= 16 - 3(4)(1) + 3(1)(1^2) - 1^3 = \boxed{6}.\end{aligned}$$

19. D Note that a given point z can be reached by either moving in the $+1$ direction from $z-1$, moving in the $+e^{i(\pi/3)}$ direction from $z - e^{i(\pi/3)}$, or moving in the $e^{i(-\pi/3)}$ direction from $z - e^{i(-\pi/3)}$. In the diagram below, this corresponds to the number of paths to one point being the sum of the three closest points to the left of it, as indicated in purple. Keeping in mind to avoid the point at 2, this gives a total of $\boxed{152}$ paths.



20. D The probability of the coin landing heads all three times is p^3, and the probability of the coin landing heads exactly two times is $\binom{3}{2}p^2(1-p)$. The given condition is equivalent to

$$\frac{p^3}{p^3 + 3p^2(1-p)} = \frac{1}{5} \rightarrow 5p = p + 3(1-p) \rightarrow p = \frac{3}{7} \rightarrow \boxed{10}.$$

21. **C** Since $\binom{n}{k} = \binom{n}{n-k}$, the desired quantity is equivalent to

$$\binom{5}{0}\binom{6}{6} + \binom{5}{1}\binom{6}{5} + \cdots + \binom{5}{5}\binom{6}{1}.$$

Consider the following counting argument: suppose you have 5 red coins and 6 blue coins, and you want to pick 6 of them. You could do this in two ways:

- If you want to pick i red coins, for $i = 0, \ldots, 5$, there are $\binom{5}{i}\binom{6}{6-i}$ ways to do so and still get 6 coins in total. Summing over all $i = 0, \ldots, 6$ gives the total number of ways to pick 6 coins (which is our desired quantity).
- Pick 6 coins out of 11 coins, for a total of $\binom{11}{6}$ ways.

Since these both count the same thing, they must be equal, so k is either 5 or 6, the larger of which is $\boxed{6}$.

Remark: *Vandermonde's identity* is a generalization of this problem.

22. **C** The expected contribution by the term $\frac{1}{n}$ is given by $\frac{1}{2^n} \cdot \frac{1}{n}$. By linearity of expectation, Jack's expected sum is $\sum_{n=1}^{\infty} \frac{1}{n \cdot 2^n}$. Note that this is $\sum_{n=1}^{\infty} \frac{x^n}{n}$ for $x = \frac{1}{2}$. We can compute

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = \sum_{n=1}^{\infty} \int x^{n-1}\, \mathrm{d}x = \int \sum_{n=1}^{\infty} x^{n-1}\, \mathrm{d}x = \int \frac{1}{1-x}\, \mathrm{d}x = -\ln(1-x).$$

Plugging in $x = \frac{1}{2}$ gives $-\ln\left(\frac{1}{2}\right) = \boxed{\ln(2)}$.

23. **B** Suppose the roots of the quadratic in question are r and s with $r \leq s$, then we can write $-x^2 + ax + b = -(x-r)(x-s)$. The area in question is given by

$$\int_r^s -(x-r)(x-s)\, \mathrm{d}x = \int_r^s -x^2 + (r+s)x - rs\, \mathrm{d}x$$

$$= \left[-\frac{x^3}{3} + \frac{(r+s)x^2}{2} - rsx\right]_r^s = -\frac{s^3 - r^3}{3} + \frac{(r+s)(s^2 - r^2)}{2} - rs(s-r)$$

$$= \frac{-2s^3 + 2r^3 + 3s^3 - 3r^3 + 3s^2r - 3r^2s - 6s^2r + 6r^2s}{6}$$

$$= \frac{s^3 - 3s^2r + 3sr^2 - r^3}{6} = \frac{(s-r)^3}{6}.$$

We can see that the area being at least 36 is equivalent to the difference between the roots being at least 6. With the quadratic formula we can compute the difference between the roots to be $\sqrt{a^2 + 4b}$, so we want $a^2 + 4b \geq 36$. Using geometric probability, this corresponds to (a, b) lying above the parabola $b = 9 - \frac{a^2}{4}$ in the square with vertices (0,0), (10,0), (10,10), and (0,10). The area below the parabola is given by $\int_0^6 9 - \frac{a^2}{4}\, \mathrm{d}a = \left[9a - \frac{a^3}{12}\right]_0^6 = 36$, so the desired probability is $1 - \frac{36}{100} = \boxed{\frac{16}{25}}$.

24. **D** The probability that it takes Louis an even number of flips to get his first head is $\frac{1}{2^2}+\frac{1}{2^4}+\frac{1}{2^6}+\cdots=\frac{1}{3}$. Now, note that flipping four heads is equivalent to flipping one head four times. In order for this to take an overall even number of flips, either 0, 2, or 4 of these heads must take an even number of flips. These occur with probability $\left(\frac{2}{3}\right)^3=\frac{16}{81}$, $\binom{4}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2=\frac{24}{81}$, and $\left(\frac{1}{3}\right)^4=\frac{1}{81}$ respectively, for a total of $\boxed{\frac{41}{81}}$.

25. **B** The expected value is given by

$$\mathbb{E}[X]=\int_0^\infty xf(x)\,\mathrm{d}x=\int_0^\infty \frac{Ax}{(1+x^4)(1+x^8)}\,\mathrm{d}x.$$

Substituting $u=x^2$ gives

$$\mathbb{E}[X]=\frac{A}{2}\int_0^\infty \frac{1}{(1+u^2)(1+u^4)}\,\mathrm{d}u.$$

Substituting $u=\frac{1}{t}$ gives

$$\mathbb{E}[X]=\frac{A}{2}\int_\infty^0 \frac{1}{(1+\frac{1}{t^2})(1+\frac{1}{t^4})}\left(-\frac{1}{t^2}\right)\mathrm{d}t=\frac{A}{2}\int_0^\infty \frac{t^4}{(1+t^2)(1+t^4)}\,\mathrm{d}t.$$

Adding gives

$$2\mathbb{E}[X]=\frac{A}{2}\int_0^\infty \frac{1+t^4}{(1+t^2)(1+t^4)}\,\mathrm{d}t=\frac{A}{2}\int_0^\infty \frac{1}{1+t^2}\,\mathrm{d}t=\frac{A}{2}[\arctan(t)]_0^\infty=\frac{A\pi}{4}.$$

Finally, dividing by 2 gives $\mathbb{E}[X]=\boxed{\frac{A\pi}{8}}$.

26. **B** Let F_0, F_H, and F_{HH} denote the probabilities that Free wins when the current sequence ends in no heads, exactly one head, or two (or more) heads respectively. Note that Agata has the same probabilities when he has the coin, so the probability that Free wins in these cases are $1-F_0$, $1-F_H$, and $1-F_{HH}$ respectively (note that the probability of never getting HHT approaches 0, so someone wins eventually). If free flips the coin with no heads at the end of the sequence, Agata gets the coin with either no heads (probability $1/2$) or one head (probability $1/2$). This gives the equation

$$F_0=\frac{1}{2}(1-F_0)+\frac{1}{2}(1-F_H).$$

Similar reasoning for one head gives

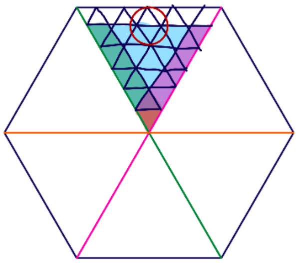
$$F_H=\frac{1}{2}(1-F_0)+\frac{1}{2}(1-F_{HH}).$$

The two heads case is a bit special, since Free either wins (probability $1/2$) or gives the coin to Agata still with two heads at the end of the sequence. This means

$$F_{HH}=\frac{1}{2}(1)+\frac{1}{2}(1-F_{HH}).$$

This is a linear equation in F_{HH} and can be solved to get $F_{HH}=\frac{2}{3}$. Plugging this in and rearranging the first two equations gives $\frac{3}{2}F_0=1-\frac{1}{2}F_H$ and $F_H=\frac{2}{3}-\frac{1}{2}F_0$. Solving this gives $(F_0,F_H)=\left(\boxed{\frac{8}{15}},\frac{2}{5}\right)$.

27. D Split the hexagon into six equilateral triangles as shown below; by symmetry, we only need to consider one triangle. We can compute that the height of one of these triangles is 6, and using the fact that the radius of the disk is 1 "pushes in" our set of possible centers by 1. This motivates splitting the triangle into 36 congruent equilateral triangles of height 1.



Note that the set of points within 1 unit of each diagonal (and thus the disk intersects the diagonal) also nicely line up with these triangles. We can then see that there is 1 triangle close to all three diagonals, 1 close to two diagonals, 14 close to one diagonal, and 9 close to 0 diagonals. There are 25 triangles total (that our center can lie in), so the probabilities of 0,1,2,3 diagonals are $\frac{9}{25}, \frac{14}{25}, \frac{1}{25}, \frac{1}{25}$ respectively. This gives an expected value of $0\left(\frac{9}{25}\right) + 1\left(\frac{14}{25}\right) + 2\left(\frac{1}{25}\right) + 3\left(\frac{1}{25}\right) = \boxed{\frac{19}{25}}$.

28. A Note that we cannot just pick a random $\theta \in [-1, 1]$ then pick a random $r \in [0, 1 - \theta^2]$ because this is not uniform as there will be more points towards the origin.

Instead, we set up a probability density function $f(r)$ for the distance from a randomly-selected point to the origin. Note that the "θ range" for a given r is $-\sqrt{1-r} < \theta < \sqrt{1-r}$, so the "arc length" for a given r is $r \cdot 2\sqrt{1-r}$. This means that $f(r) = kr\sqrt{1-r}$ for appropriate constant k. We can integrate

$$\int_0^1 r\sqrt{1-r} = \int_0^1 (1-r)\sqrt{r}\,\mathrm{d}r = \int_0^1 r^{1/2} - r^{3/2}\,\mathrm{d}r = \frac{4}{15},$$

So $k = \frac{15}{4}$. We can now compute

$$\mathbb{E}[r] = \int_0^1 rf(r)\,\mathrm{d}r = \int_0^1 \frac{15}{4}r^2\sqrt{1-r}\,\mathrm{d}r = \frac{15}{4}\int_0^1 r^{1/2} - 2r^{3/2} + r^{5/2}\,\mathrm{d}r = \boxed{\frac{4}{7}}.$$

29. B For $k = 0, \ldots, n$, the probability that Durham and Miguno both end up at $(k, n-k)$ is $\left(\binom{n}{k} \cdot \frac{1}{2^n}\right)^2 = \frac{\binom{n}{k}^2}{4^n}$. This means the probability that Durham and Miguno end up at the same point is $P(n) = \sum_{k=0}^{n} \frac{\binom{n}{k}^2}{4^n}$. Using the same method as in problem 21, we can compute $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$, so $P(n) = \frac{\binom{2n}{n}}{4^n} = \frac{\frac{(2n)!}{n!n!}}{4^n}$. By Stirling's approximation, as n grows large

$$P(n) \approx \frac{\sqrt{2\pi(2n)}\left(\frac{2n}{e}\right)^{2n}}{4^n\left(\sqrt{2\pi n}\right)^2\left(\left(\frac{n}{e}\right)^n\right)^2} = \frac{1}{\sqrt{\pi n}}$$

In order for $\lim_{n\to\infty} n^r P(n)$ to exist and be nonzero, we must have $r = \boxed{\frac{1}{2}}$.

30. **A** Plugging in $r = \frac{1}{2}$ in the previous setup gives

$$\lim_{n\to\infty} n^r\, P(n) = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{\pi n}} = \boxed{\frac{1}{\sqrt{\pi}}}.$$