1. E We need to check both the highest power of two that divides 2022! and the highest power of 3. We can compare half the highest power of two (which makes 4) that divides 2022! and the highest power of 3 that divides 2022!. 1006 is the highest power of 3, and 1007 is the highest power of 4, which means that our answer is the smaller of the two, **1006.**
2. B It is easy enough to count the cases. The first case is if the three colors are the same. There are 4 cases there. The second is if 2 colors are the same, and 1 is different. We choose one of the four colors to be the color for the 2 circles and choose one of the remaining three for the other. After we choose the colors, the arrangements are identical, as rotations and reflections are considered identical. That yields $4 \cdot 3 = 12$ cases. Finally, there is the case where all 3 colors are different. We choose 3 out of the 4 colors to be represented. Again, once we choose the colors, all the arrangements are the same, giving us 4 cases. Adding the cases up, we get $4 + 12 + 4 = \mathbf{20.}$
3. B The key to this question is to reflect our farmer's location over the line that represents the river, and then to find the distance between the new location of the farmer and the squirrel. Find the equation of the line passing through (1,4) with slope perpendicular to the river, or 1. that equation is $y = x + 3$. The intersection of this and $y = -x + 1$ is at point $(-1,2)$. since $(-1,2)$ is 2 units down and 2 units to the left of farmer Sam's original point, his new point is going to be 2 units down and 2 units to the left of this point, which is at $(-3,0)$. Now, the answer becomes the square of the distance between this point and the squirrel, which is $\left(6-(-3)\right)^2 + (5-0)^2 = \mathbf{106.}$
4. D We can cube $\sin x + \cos x$ on both sides, to get
$$\sin^3 x + \cos^3 x + 3\sin x\cos x\,(\sin x + \cos x) = \frac{125}{64}.$$
Squaring $\sin x + \cos x$, we get
$$\sin^2 x + \cos^2 x + 2\sin x\cos x = \frac{25}{16},$$
$$1 + 2\sin x\cos x = \frac{25}{16},$$
$$\sin x\cos x = \frac{9}{32}.$$
Plugging in our known values to the cubed equation, we get
$$\sin^3 x + \cos^3 x + 3\left(\frac{9}{32}\right)\left(\frac{5}{4}\right) = \frac{125}{64},$$
$$\sin^3 x + \cos^3 x = \mathbf{\frac{115}{128}}.$$
5. C We can treat this as an infinite geometric series since the magnitude of the common difference is less than 1.
$$S = \frac{a}{1-r};$$
$$S = \frac{4}{1 - \frac{1+2i}{3}} = \frac{12}{2-2i}.$$
The magnitude of our answer is then $\frac{12}{2\sqrt{2}} = \mathbf{3\sqrt{2}}.$

6. C Call the roots r_1, r_2, and r_3, . We know that $f(p) = f(q) = f(r) = 0$. Using this, we obtain 3 separate equations:

$$r_1^3 - 7r_1^2 - 9r_1 + 12 = 0,$$
$$r_2^3 - 7r_2^2 - 9r_2 + 12 = 0,$$
$$r_3^3 - 7r_3^2 - 9r_3 + 12 = 0.$$

Adding all three equations, we get:

$$\sum r_n^3 - 7\sum r_n^2 - 9\sum r_n + 36 = 0,$$

Where $\sum$ represents the sum of the roots.

Isolating what we want on the left-hand side, we can solve for the sum of the cube of the roots in terms of the sum of squares, the sum, and a constant. These are easy to solve using Vieta's formulas and manipulation.

$$\sum r_n^2 = \left(\sum r_n\right)^2 - 2\sum_{cycl} r_a r_b = \left(\frac{b}{a}\right)^2 - 2\left(\frac{c}{a}\right);$$
$$\sum r_n = \frac{-b}{a}.$$

From these formulas, we can solve for each of the unknowns. we get that $\sum r_n^2 = 67$ and $\sum r_n = 7$. Substituting into our equation, we get

$$\sum r_n^3 = 7 \cdot 67 + 9 \cdot 7 - 36 = \mathbf{496}.$$

7. A Let the midpoints of AH, BH, and CH be D, E, and F, respectively. Notice that this triangle is the result of a homothety of center H and factor $\frac{1}{2}$. In other words, triangle DEF is similar to ABC, with orthocenter at H and $\frac{DH}{AH} = \frac{1}{2}$. This means that the circumradius of the triangle DEF is just half of the circumradius of triangle ABC. the circumradius of ABC can be calculated as $R = \frac{abc}{4[ABC]} = \frac{65}{8}$, which means that our answer is $\frac{1}{2}R = \frac{65}{16}$, or $\mathbf{81}$.

8. C x and y being positive tells us that AM-GM should be used. we know that

$$\frac{3x + 2y}{2} \geq \sqrt{3x \cdot 2y};$$
$$8 \geq \sqrt{6xy};$$
$$xy \leq \frac{64}{6} = \frac{\mathbf{32}}{\mathbf{3}}.$$

9. D This is also AM-GM, with a few extra steps. the final product weights x three times as much as y. Since the equality case of AM-GM requires all elements to be equal, we need to split our $3x$ in the known equation up into three equal components of x, and use AM-GM on the resulting 4 elements:

$$\frac{x + x + x + 2y}{4} \geq \sqrt[4]{x \cdot x \cdot x \cdot 2y};$$
$$4 \geq \sqrt[4]{2x^3y};$$
$$\mathbf{128} \geq x^3y.$$

10. C We start with the most restrictive dog breed, the dachshund. choose 2 out of the 3 rows to have a dachshund and choose one of the 3 seats in each row for each of the 2 dachshunds. For the 7 remaining seats, choose 3 of them to be for border collies. Our final answer is

$$\binom{3}{2} \cdot 3^2 \cdot \binom{7}{3} = \mathbf{945}.$$

11. D Because x and y are interchangeable in both what we're minimizing and the equation itself, we can assume that x and y are positive, maximize xy, and negate it. Maximizing requires a simple AM-GM:

$$\frac{x^2 + y^2}{2} \geq \sqrt{x^2 y^2}.$$

This means that xy is maximized at 25 for positive values of x and y, Which means that the minimum value is just the negation, or $\mathbf{-25.}$

12. E Making a quick sketch, it's clear that $f(x)$ intersects $f^{-1}(x)$. The point of intersection is then the closest point on $f(x)$ to $f^{-1}(x)$ as the distance is 0. The point can be found by equating $f(x)$ to x, giving us $x - 6 = x^3$, or $x = \mathbf{-2}$.

13. B Our equation, interpreted in the argand plane, is the locus of all points where the sum of the distances from it to $(6, 0)$ and $(0, -8)$ is 26. This is the definition of an ellipse. In this ellipse, the distance between the two points represent $2c$, and the sum of the distances represent $2a$. We know that $a^2 = b^2 + c^2$ and that the area of an ellipse is πab. From the information we have, we know that $2a = 26;\ a = 13$ and $2c = 10; c = 5$, which follows that $b = 12$. this means that the area of the ellipse is $\pi \cdot 13 \cdot 12 = \mathbf{156\pi}$.

14. A Solve for each of the entries first:
1. the first 3 pentagonal numbers are 1, 5, 12, found by taking the differences of the differences of the first three terms and setting it equal to 3. **(12)**
2. The formula for the sum of the first n squares is $\frac{n(n+1)(2n+1)}{6}$ (alternatively, just add them up). **(30)**
3. The formula for a perfect number is $(2^p - 1)(2^{p-1})$, where p and $2^p - 1$ are prime. $p = 3$ yields 28, and $p = 5$ yields 496, which means that the biggest perfect number less than 100 is **28**. **(28)**
4. the first 6 triangular numbers are 1, 3, 6, 10. Summing them up, we get **20. (20).**
We maximize the determinant of a 2×2 matrix with only positive entries by putting the biggest elements along its diagonal. our answer is therefore $30 \cdot 28 - 12 \cdot 20 = \mathbf{600}$.

15. B Since 3, 5, and 7 are relatively prime, we can multiply our result from just 3 and 5 by $\frac{6}{7}$. There are $\frac{2100}{3} + \frac{2100}{5} - \frac{2100}{15} = 980$ numbers that are divisible by 3 or 5. out of these, $\frac{6}{7}$ are also not divisible by 7. multiplying by $\frac{6}{7}$, we obtain **840**.

16. B Checking the finite differences, and the third finite difference is 6, indicating a cubic fits the given points.

Specifically, the points can be plugged into a generic cubic, and the system can be solved to yield $f(x) = x^3 - 3x^2 + 5x - 2$. Note that the third finite difference being $6 = 3!$ gives a leading coefficient of 1, which speeds up the process. The function can then be integrated to produce a result of 32.

However, since Simpson's Rule is precise for a cubic, it is sufficient to compute the integral by doing Simpson's Rule with 2 subintervals, yielding

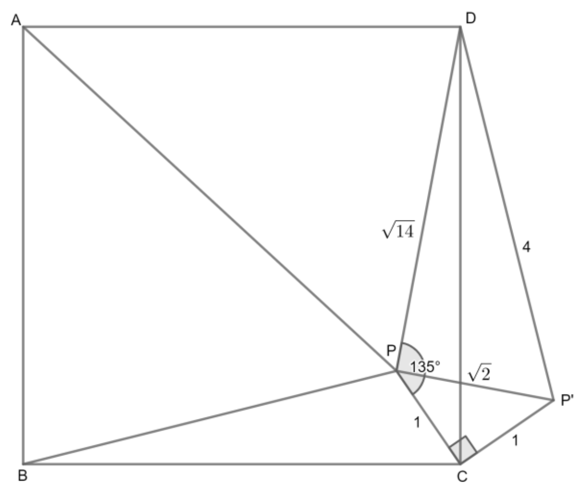
$$\frac{1}{3} \cdot 4(1 \cdot (-2) + 4 \cdot 4 + 1 \cdot 34) = \mathbf{32}$$

17. B Notice that between two integers, the left-hand side of the equation is monotonically increasing, while the right-hand side stays constant. Also notice that the left-hand side goes to 0 at every integer value and gets bigger than the right-hand side at some point between every integer value. This means that for every interval between two integers, we will have a value of x that satisfies this equation. Since there are 2021 intervals between 1 and 2022, our final answer is **2021.**

18. B There are 3 pairs of same letters: L, I, and E. We first count the number of ways to arrange OLIVIALEE, ignoring restrictions. Since the Ls, Is and Es are indistinguishable, we have $\frac{9!}{2^3}$. Now, take out the cases when the Ls are together, the Is are together, and the Es are together. Each of these cases yield $\frac{8!}{2^2}$ cases. However, when we count these, we undercount the cases where 2 of the 3 pairs are together. Each of those yield $\frac{7!}{2}$ cases. Finally, we have overcounted the case where all 3 pairs are together by 1, which means we must subtract 6!. Our final answer is

$$\frac{9!}{8} - 3 \cdot \frac{8!}{4} + 3 \cdot \frac{7!}{2} - 6! = \mathbf{21960}.$$

19. C By the British Flag theorem, we know that $BP^2 + DP^2 = AP^2 + CP^2$. plugging in the values we have, we get $AP = \sqrt{BP^2 + DP^2 - CP^2} = \sqrt{16 + 14 - 1} = \sqrt{\mathbf{29}}$.



20. D Rotate triangle BPC clockwise 90° about point C. We are motivated to do so by what the question is asking for and by the hint. The resulting image is shown above. Since $\overline{PC} = \overline{P'C} = 1$, and since $\angle P'CD + \angle DCP = \angle PCB + \angle PCD = 90°$, triangle $PP'C$ is an isosceles right triangle. Notice that by the Pythagorean theorem, DPP' is a right triangle with the right angle at P. This means that $\angle DPC = 45° + 90° = 135°$. Now we have $[DPC] = \frac{1}{2} \cdot \overline{DP} \cdot \overline{PC} \cdot \sin(135°) = \frac{1}{2} \cdot \sqrt{14} \cdot 1 \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{\mathbf{7}}}{\mathbf{2}}$.

21. A It is easy enough to count the cases. Start from A. There are 3 possible vertices that a road can go to. B and D only yield 1 path each. P yields two paths, which means that the final answer is $1+1+2=\mathbf{4}$.

22. D The answer is 0. For every case where $C-A=k$, where k is a positive integer, there is a case for which $C-A=-k$. These all sum to 0, making the expected value 0.

23. A We can calculate the probability of every difference occurring, up to a difference of $8\ (9-1)$. The lowest the difference can be is 2, where the three numbers are consecutive. This happens with probability $\frac{7}{84}=\frac{1}{12}$. 3 can be done with probability $6*\frac{2}{84}=\frac{1}{7}$. In general, the probability can be expressed as $\frac{1}{84}\sum_{n=1}^{7}(8-n)(n)=8n-n^2$. Since $E=\sum xp(x)$, $E=\frac{1}{84}\sum_{n=1}^{7}(n+1)(8n-n^2)=8n+7n^2-n^3$. Splitting up the sum, and solving, we get $\frac{1}{84}\left(\frac{8(7)(8)}{2}+\frac{7(7)(8)(15)}{6}-(7)^2(4)^2\right)=\frac{15}{3}=\mathbf{5}$.

Alternate solution:

After listing out the probabilities, one could realize that it is symmetric around 5, which makes the answer **5.**

24. B

$$\frac{\left(a^2-3^2-c^2-6c\right)\left(a^2-3^2-c^2+6c\right)}{\left[\left(a-3\right)^2-c^2\right]\left[\left(a-c\right)\left(a+c\right)+3\left(a+c\right)\right]}=\frac{\left[a^2-\left(c^2+6c+9\right)\right]\left[\left[a^2-\left(c^2-6c+9\right)\right]\right]}{\left(a-3-c\right)\left(a-3+c\right)\left(a+c\right)\left(a-c+3\right)}$$

$$\frac{\left(a-c-3\right)\left(a+c+3\right)\left(a+c-3\right)\left(a-c+3\right)}{\left(a-3-c\right)\left(a-3+c\right)\left(a+c\right)\left(a-c+3\right)}=\frac{a+c+3}{a+c}$$

25. B Draw yourself a picture:

$$b+h=31\rightarrow b=31-h\rightarrow 2x+b=50\rightarrow x=25-\frac{b}{2}\rightarrow h^2=x^2-\left(\frac{b}{2}\right)^2$$

$$h^2=\left(25-\frac{b}{2}\right)^2-\left(\frac{b}{2}\right)^2=625-25b=625-25\left(31-h\right)$$

$$h^2-25h+150=0\rightarrow\left(h-10\right)\left(h-15\right)=0\rightarrow 25$$

26. B Draw rectangle $WXYZ$ such that S and Q are on WX, U is on XY, R and A are on YZ, and E is on ZW. Triangles $EWS, QXU, UYA,$ and RZE are all $30-60-90$ triangles. We know that XU= $3\sqrt{3}$ and $UY=4\sqrt{3}$, so $XY=WZ=7\sqrt{3}$. We know $WS=5$, $QX=3$, so $WZ=12$. We know that $WE=5\sqrt{3}$ so $EZ=7\sqrt{3}-5\sqrt{3}=2\sqrt{3}$. The area of the rectangle is $7\sqrt{3}\cdot 12=84\sqrt{3}$, and the sum of the areas of the right triangles are $5\sqrt{3}\cdot\frac{5}{2}+3\sqrt{3}\cdot\frac{3}{2}+2\sqrt{3}\cdot\frac{2}{2}+4\sqrt{3}\cdot\frac{4}{2}=27\sqrt{3}$. Subtracting this from the rectangle, we get the answer of $\mathbf{57\sqrt{3}}$.

27. D Using product of chord segments you get $6\bullet 24=12\bullet 12$. So the longest chord is the diameter at 30 and the shortest chord would be24.There is only 1 shortest chord and 1 longest chord but chords of length 25,26,27,28, and 29 can occur twice. That is a total of 12

28. B Let A be (a, a^2) and B be (b, b^2). Applying shoelace, $[BOA] = \frac{1}{2}|a^2b - ab^2|$. The selection of a and b is on a 6×6 square. Since a and b are interchangeable, it is sufficient to consider the case $b > a$, which is random uniform distribution over a 6×6 right triangle. This means the probability density function is simply $\frac{1}{18}$.

So the expected value of $[BOA]$ can be computed as below:

$$\frac{1}{18}\int_0^6 \int_0^b \frac{1}{2}(ab^2 - a^2b)da\,db = \frac{1}{36}\int_0^6 \left(\frac{1}{2}a^2b^2 - \frac{1}{3}a^3b\right)\Big|_0^b db$$
$$= \frac{1}{216}\int_0^6 b^4\,db = \frac{1}{216}\cdot\frac{1}{5}b^5]_0^6 = \mathbf{\frac{36}{5}}$$

29. A Listing out the unit's digit of the sequence is the easiest way to approach this problem, because it is quite easy to realize that the units digit cycles every 4. The cycle starting from 0, is 2, 6, 4, 0, 2, 6, 4, 0, …. The 2023rd term will then be **0.**

Alternate solution:

We could also find the explicit form of this recurrence. This is our standard linear recurrence, but with a twist: the constant. To get rid of this constant, we shift the indices by 1, and subtract the two equations. First, move every term over to the left side:

$$A_n - 5A_{n-1} + 6A_{n-2} = 6.$$

Shift every index by 1:

$$A_{n-1} - 5A_{n-2} + 6A_{n-3} = 6.$$

Subtract the two, combining like terms:

$$A_n - 6A_{n-1} + 11A_{n-2} - 6A_{n-3} = 0.$$

This is our standard linear recurrence, with explicit solution

$$A_n = \alpha r_1^n + \beta r_2^n + \gamma r_3^n,$$

With r_1, r_2, r_3 representing the roots of the characteristic polynomial. Our characteristic polynomial is found by assuming that A_n is geometric and solving for the roots of the resulting polynomial in terms of the common difference. The roots of our polynomial are 1,2, and 3. Using our initial values, and finding A_2, and plugging into this form, we get

$$12 = \alpha + \beta + \gamma,$$
$$26 = \alpha + 2\beta + 3\gamma,$$
$$64 = \alpha + 4\beta + 9\gamma.$$

Solving for α, β, and γ, we get $\alpha = 3, \beta = 4, \gamma = 5$. plugging in 2023 for n, we get

$$A_{20} = 3 + 4 \cdot 2^{2023} + 5 \cdot 3^{2023}.$$

2^{2023} ends in 8, so $4 \cdot 2^{2023}$ ends in 2. Since $5 \cdot 3^{2023}$ is odd, and since it's a multiple of 5, it ends in 5. Our final answer is $3 + 2 + 5(mod\ 10) = \mathbf{0}$.

30. C Claim: At time $2t + 1$ for integer t, Jeremy will be at a point directly adjacent to his original position.

Proof: index the points so that Jeremy's original point is 0, the points adjacent are both 1, and the point opposite him is 0. Notice that the points that are adjacent to each

point is of opposite index. Since Jeremy starts at index 0, he must be at index 1 for odd moves, and at index 0 for even moves.
Since Jeremy will be at index 1 after 2021 seconds, at 2022 seconds he has a $\frac{1}{2}$ probability of being at his starting point, and a $\frac{1}{2}$ probability at being at the opposite point.