All uppercase letter variables are positive integers unless otherwise stated. All fractions containing uppercase letter variables are in lowest terms. NOTA means "None of the Above."

~~~~~~ Good luck, and have fun! ~~~~~~

1. Let $f(x) = x^2 + 4x + n$. If $f(k) = f'(k) = f''(k)$, find $k + n$.

    A. -4 B. -2 C. 2 D. 4 E. NOTA

2. For some $\frac{\pi}{2} < \theta < \frac{3\pi}{4}$, $\sin^4 \theta + \cos^4 \theta = \frac{7}{8}$. Find $\sin(4\theta)$.

    A. $\dfrac{1}{2}$ B. $\dfrac{1}{\sqrt{3}}$ C. $\dfrac{1}{\sqrt{2}}$ D. $\dfrac{\sqrt{3}}{2}$ E. NOTA

3. Evaluate: $\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^{2023}$.

    A. $-\dfrac{1}{2} - \dfrac{i\sqrt{3}}{2}$ B. $-\dfrac{1}{2} + \dfrac{i\sqrt{3}}{2}$ C. $\dfrac{1}{2} - \dfrac{i\sqrt{3}}{2}$ D. $\dfrac{1}{2} + \dfrac{i\sqrt{3}}{2}$ E. NOTA

4. Find the sum of the possible values of x such that $\begin{bmatrix} 2 & 1 & -2 \\ x+2 & |x| & -4 \\ 3 & x+15 & x \end{bmatrix}$ is singular.

    A. $-\dfrac{144}{5}$ B. -22 C. $-\dfrac{34}{5}$ D. -6 E. NOTA

5. Identify the conic section represented by $x^2 + 2y^2 + 4xy + 8x + 16y = 32$.

    A. Ellipse   B. Hyperbola   C. Parabola   D. Degenerate   E. NOTA

6. Let the roots of $x^3 + 6x^2 - 8x + 12$ be r_1, r_2, and r_3. Find $(r_1 + r_2)(r_2 + r_3)(r_3 + r_1)$.

    A. -60 B. -36 C. 36 D. 60 E. NOTA

7. Let $f(x) = x^3 + 4x + 5$. Find the slope of the line tangent to $f^{-1}(x)$ at its y-intercept.

    A. $\dfrac{1}{79}$ B. $\dfrac{1}{52}$ C. $\dfrac{1}{7}$ D. $\dfrac{1}{4}$ E.   NOTA

8. Luke's Legumes can sell 80 beans for \$45 each. Jun's market research shows that for every \$2.50 reduction in the price, Luke can sell 10 additional beans. If Luke adjusts the price of a bean to maximize his revenue, how much more revenue would he make?

    A.   \$450   B.   \$500   C.   \$625   D.   \$750   E.   NOTA

9. Evaluate $\lim\limits_{x \to 0} \left(\dfrac{1}{x^2} - \dfrac{e^x}{(e^x - 1)^2} \right)$.

    A. $\dfrac{1}{12}$ B. $\dfrac{1}{9}$ C. $\dfrac{1}{8}$ D. $\dfrac{1}{6}$ E.   NOTA

10. Let $f(x^3) = \int_{1-x^2}^{e^x} (12t^3 + 6t)\, dt$. If $f'(1) = Ae^C + Be^D$, find $A + B + C + D$.

    A.   12   B.   16   C.   20   D.   24   E.   NOTA

    For questions 11 and 12, $f(x) = x^2 + 4x + 8$.

11. Alan is standing at the point $(3,3)$, and John is standing somewhere on the graph of the function $f(x)$. Find the shortest possible distance between them.

    A. $3\sqrt{2}$ B. $2\sqrt{5}$ C. $2\sqrt{6}$ D. 5 E.   NOTA

12. Srijan wants to hit the bullseye of a strange dartboard. The dartboard can be represented by the circle $x^2 + y^2 + 4x - 12y + 36 = 0$, and the bullseye is the region bounded by $f(x)$ and its latus rectum. Given that Srijan throws a dart that lands in a random location on the dartboard, find the probability Srijan throws a bullseye on his first attempt.

    A. $\dfrac{1}{48\pi}$ B. $\dfrac{1}{36\pi}$ C. $\dfrac{1}{24\pi}$ D. $\dfrac{1}{18\pi}$ E.   NOTA

13. Evaluate: $\int_0^{\pi/2} \sin^6 \theta \cos^3 \theta \, d\theta$.

   A. $\dfrac{1}{40}$ B. $\dfrac{2}{63}$ C. $\dfrac{1}{9}$ D. $\dfrac{11}{63}$ E. NOTA

14. One of Carolina's many fun and wacky Zoom background images displays a bunch of implicitly defined functions. Bored in Mr. Birenbaum's class, Rosa picks one of them and decides to find the slope of its tangent line at its positive x-intercept. If she picks $(x^2 + y^2 + 2y)^2 = 4(x^2 + y^2)$, what right answer would she obtain?

   A. -2 B. -1 C. $-\dfrac{1}{2}$ D. $-\dfrac{1}{4}$ E. NOTA

15. Let $\mathbb{S}_n(f(x))$ equal the sum of the coefficients of the degree-n Maclaurin series for $f(x)$. Find $\left\lfloor \lim_{n\to\infty} \mathbb{S}_n(e^x \arctan x) \right\rfloor$, where $\lfloor x \rfloor$ is the greatest integer function.

   A. 1   B. 2   C. 3   D. 4   E. NOTA

16. If $\int_{-3}^4 \dfrac{x^3}{\sqrt{16+x^2}} \, dx = \dfrac{A - B\sqrt{C}}{D}$ for squarefree C, find $A + B + C + D$.

   A. 130   B. 184   C. 197   D. 210   E. NOTA

17. Evaluate: $\lim_{x\to 0^+} (\ln x \cdot \tan(2x))$.

   A. -2 B. -1 C. $-\dfrac{1}{2}$ D. 0 E. NOTA

18. The relation $f(x,y)$ satisfies the relation $x^2 \dfrac{dy}{dx} = \dfrac{3x^2+1}{3y^2+1}$ for $x > 0$ and is not defined for $x < 0$. If $f(1,1) = 0$, find the value of k such that $f(k,0) = 0$.

   A. $\dfrac{1}{3}$ B. $\dfrac{1}{\sqrt{3}}$ C. 1   D. $\sqrt{3}$ E. NOTA

19. A conic has equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. The coordinate axes are rotated by an angle of θ to form the conic $A'x^2 + B'xy + C'y^2 + D'x + E'y + F = 0$. Which of the following statements regarding invariants can be false? Note: $0 < \theta < \frac{\pi}{2}$.

A. $A + C = A' + C'$

B. $B^2 - 4AC = B'^2 - 4A'C'$

C. $D^2 + E^2 = D'^2 + E'^2$

D. $\begin{vmatrix} A & B/2 & D/2 \\ B/2 & C & E/2 \\ D/2 & E/2 & F \end{vmatrix} = \begin{vmatrix} A' & B'/2 & D'/2 \\ B'/2 & C' & E'/2 \\ D'/2 & E'/2 & F \end{vmatrix}$

E. NOTA

20. A relation can be parametrized by $\langle x, y \rangle = \langle 2t^2 - 3t + 2, t^2 + 6t + 8 \rangle$. Find $\left. \frac{d^2 y}{dx^2} \right|_{\langle x,y \rangle = \langle 7,3 \rangle}$.

A. $-\frac{4}{7}$ B. $-\frac{30}{49}$ C. $\frac{4}{49}$ D. $\frac{30}{343}$ E. NOTA

21. Find the slope of the line normal to $f(x) = x^3 - 6x^2 + 9x$ at its inflection point.

A. $\frac{1}{3}$ B. $\frac{1}{6}$ C. $\frac{1}{9}$ D. $\frac{1}{12}$ E. NOTA

22. An upright cone is inscribed in an inverted cone with radius 5 and height 20. Let the maximum volume of the smaller cone be $\frac{A\pi}{B}$. Find the number of positive integer factors of the product AB.

A. 72   B. 80   C. 100   D. 144   E. NOTA

23. Functions $f(x)$ and $g(x)$ satisfy the following system. If $f(0) = 1$ and $g(0) = 3$, find e^k, where k is the unique value such that $f(k) = g(k)$.
$$5f'(x) = 9f(x) - 3g(x)$$
$$5g'(x) = 2f(x) + 16g(x)$$

A. $\frac{2}{3}$ B. $\frac{3}{4}$ C. $\frac{4}{3}$ D. $\frac{3}{2}$ E. NOTA

24. Find the area of the inner loop of the graph of the polar curve $r = 4 + 8\cos\theta$.

    A. $48\sqrt{3} - 24\pi$ B. $6\pi - 6\sqrt{3}$ C. $16\pi - 24\sqrt{3}$ D. $16\sqrt{3} - 6\pi$ E. NOTA

25. Evaluate: $8092 \int_0^{\pi/2} \frac{\sin^{2023}\theta}{\sin^{2023}\theta + \cos^{2023}\theta} \, d\theta$.

    A. 2π B. 4π C. 2023π D. 4046π E.   NOTA

26. Evaluate: $\int_0^{\pi/2} \frac{\sin\theta}{\sin^3\theta + \cos^3\theta} \, d\theta$.

    A. $\dfrac{\pi}{3\sqrt{3}}$ B. $\dfrac{2\pi}{3\sqrt{3}}$ C. $\dfrac{5\pi}{6\sqrt{3}}$ D. $\dfrac{\pi}{\sqrt{3}}$ E.   NOTA

    Hint: By Weierstrass, $t = \tan\frac{\theta}{2}$, giving $\int f(\sin\theta, \cos\theta) \, d\theta = \int \frac{2}{1+t^2} f\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right) dt$.

27. Find the sum of the digits of $\left\lfloor 100 \sum_{n=1}^{\infty} \text{arccot}\left(\frac{n^2}{2}\right) \right\rfloor$.

    A.   10   B.   11   C.   12   D.   13   E.   NOTA

28. Evaluate: $\int_1^{\infty} e^{-x^2} \, d(x^2 + 1)$.

    A. $2e$ B. e C.   2   D.   1   E.   NOTA

29. Rithvik has mounted a 16-ft tall screen 9 feet off the ground against a vertical wall. How far away in feet should Anagh (who is very short just like a point on the ground) sit if he wants to maximize his viewing angle between the base of the screen and the top of the screen?

    A.   12   B.   15   C.   18   D.   20   E.   NOTA

30. Clayton decides to stop teasing Anagh for being short and instead wants to find the answer to Life, The Universe, and Zeverything. He knows that the answers to these concepts are at the points $L = (-1, -1)$, $U = (5,8)$, and $Z = (3,19)$. Find the area of triangle LUZ.

    A.   42   B.   48   C.   54   D.   72   E.   NOTA