DDCDB DCCAA BCBBB BDBED ACACC BADBA

1. D $f'(k) = 2k + 4$ and $f''(k) = 2$. Setting these equal gives $k = -1$. $f'(-1) = 2$. $f(-1) = 1 - 4 + n = n - 3 = 2$, so $n = 5$. $k + n = 4$.
2. D $\sin^4\theta + \cos^4\theta = \sin^4\theta + 2\sin^2\theta\cos^2\theta\cos^4\theta - 2\sin^2\theta\cos^2\theta =$ $(\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta = 1 - 2\sin^2\theta\cos^2\theta = 1 - \frac{\sin^2(2\theta)}{2} = 1 - \frac{1-\cos(4\theta)}{4} = \frac{3}{4} + \frac{\cos(4\theta)}{4}$. Setting this equal to $\frac{7}{8}$ gives $\cos(4\theta) = \frac{1}{2}$, so with θ in the given range, $\sin 4\theta = \frac{\sqrt{3}}{2}$.
3. C $\frac{1}{2} - \frac{i\sqrt{3}}{2} = \text{cis}\left(-\frac{\pi}{3}\right)$. By de Moivre's theorem, this taken to the power of 2023 would be $\text{cis}\left(-\frac{2023\pi}{3}\right)$. Note that 2022 is a multiple of 6, so $\text{cis}\left(-\frac{2022\pi}{3}\right) = \text{cis}(2\pi) = 1$ and $\text{cis}\left(-\frac{2023\pi}{3}\right) = \text{cis}\left(-\frac{\pi}{3}\right) = \frac{1}{2} - \frac{i\sqrt{3}}{2}$.
4. D The determinant of the matrix is $2x|x| - 12 - 2(x+2)(x+15) + 6|x| - x(x+2) + 8(x+15)$. When $x \geq 0$, this is equal to $-x^2 - 22x + 48 = -(x+24)(x-2)$, so $x = 2$ is a valid root. When $x < 0$, this is equal to $-5x^2 - 34x + 48 = -(5x-6)(x+8)$, so $x = -8$ is a valid root. $x = -24$ and $x = \frac{6}{5}$ are extraneous, so the sum of the values that make the determinant 0 is -6.
5. B $B^2 - 4AC = 168 = 8$, so the conic section is putatively a hyperbola. The determinant of the degeneracy matrix is nonzero, so it is non-degenerate.
6. D By Vieta's, $r_1 + r_2 + r_3 = -6$, so the product is $(-6 - r_3)(-6 - r_2)(-6 - r_1)$. This expands to $-(abc + 6(ab + bc + ca) + 36(a + b + c) + 216) =$ $-(-12 + 6(-8) + 36(-6) + 216) = 60$.
7. C $f^{-1}(x)$ has a y-intercept where $f(x)$ has an x-intercept. By inspection, this is at $x = -1$. The slope of the tangent line to $f^{-1}(x)$ at $(0, -1)$ is $\frac{1}{f'(-1)} = \frac{1}{3x^2+4}\Big|_{x=-1} = \frac{1}{7}$.
8. C Luke's revenue as a function of the number of price reductions is $(45 - 2.5x)(80 + 10x) = -25(x - 18)(x + 8)$. The apex of this parabola is at the average of its roots, which is $x = 5$. $32.5 \cdot 130 - 45 \cdot 80 = 4225 - 3600 = 625$.
9. A Using Maclaurin series, $\frac{1}{x^2} - \frac{e^x}{(e^x-1)^2} = \frac{1}{x^2} - \frac{1}{e^x + e^{-x} - 2} = \frac{1}{x^2} - \frac{1}{2\left(\frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots\right)}$.
Common denominator produces $\frac{2\left(\frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots\right) - x^2}{2x^2\left(\frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots\right)} = \frac{2\left(\frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots\right)}{2\left(\frac{x^4}{2!} + \frac{x^6}{4!} + \frac{x^8}{6!} + \cdots\right)} = \frac{\frac{1}{4!} + \frac{x^2}{6!} + \frac{x^4}{8!} + \cdots}{\frac{1}{2!} + \frac{x^2}{4!} + \frac{x^4}{6!} + \cdots}$.
Taking a limit as x approaches 0 eliminates all but the constant terms, and the limit equals $\frac{1/24}{1/2} = \frac{1}{12}$.
10. A Using the Chain Rule on the LHS and the Second Fundamental Theorem of Calculus on the RHS, $3x^2 f'(x^3) = e^x(12e^{3x} + 6e^x) + 2x\big(12(1 - x^2)^3 + 6(1 - x^2)\big)$. Setting $x = 1$ cancels out the second term and yields $3f'(1) = 12e^4 + 6e^2$, or $f'(1) = 4e^4 + 2e^2$.
11. B Observe that $f(x) = (x + 2)^2 + 4$, having a vertex at $(-2,4)$. The parabola and the point can be translated so that the parabola becomes $f(x') = x'^2$, finding the shortest distance to $(5, -1)$. The slope of $f(x')$ at (a, a^2) is $2a$, so the equation of the

tangent line is $y = 2ax' - a^2$. The line connecting $f(x')$ to $(5, -1)$ would be normal to this line and thus have slope $-\frac{1}{2a}$ and equation $y' = \frac{5-x'}{2a} - 1$. Solving $a^2 = \frac{5-a}{2a} - 1$ yields $2a^3 + 3a - 5 = 0$, which has a root at 1 by inspection. The distance from $(1,1)$ to $(5, -1)$ is $2\sqrt{5}$.

12. C Completing the square, the circle is $(x+2)^2 + (y-6)^2 = 4$, which is centered at $(-2,6)$ having radius 2 and area 4π. Note that the nadir of the circle is $(-2,4)$, the vertex of $f(x)$. Translating again, the circle becomes $x'^2 + (y'-2)^2 = 4$. For the parabola, $p = \frac{1}{4}$ and the latus rectum lies on $y' = \frac{1}{4}$, corresponding to $x' = \pm\frac{1}{2}$. The area of the region is $\int_{-1/2}^{1/2}\left(\frac{1}{4} - x^2\right)\,dx = \int_0^{1/2}\left(\frac{1}{2} - 2x^2\right)\,dx = \frac{1}{4} - \frac{2}{3}\cdot\frac{1}{8} = \frac{1}{6}$. The probability Srijan hits the bullseye is $\frac{1/6}{4\pi} = \frac{1}{24\pi}$.

13. B Since $\cos^2\theta = 1 - \sin^2\theta$, a manipulation can lead to a $u = \sin\theta$ substitution into a polynomial. $\int_0^{\pi/2}\sin^6\theta\cos^3\theta\,d\theta = \int_0^{\pi/2}\sin^6\theta\,(1-\sin^2\theta)\cos\theta\,d\theta = \int_0^1(u^6 - u^8)\,du = \frac{1}{7} - \frac{1}{9} = \frac{2}{63}$.

14. B If $y = 0$, then $x^4 = 4x^2$, or $x = 2$. Deriving, $2(2x + 2yy' + 2y')(x^2 + y^2 + 2y) = 8x + 8y'$. Plugging in values yields $8(4 + 2y') = 16$, or $y' = -1$.

15. B Like similar problems using summation of coefficients of multinomial expansions, $\lim_{n\to\infty}\mathbb{S}_n\big(f(x)\big) = f(1)$, since a Maclaurin series is just an increasingly accurate approximation of $f(x)$. Here, $f(1) = \frac{\pi e}{4}$. Using $\pi \approx 3.1$ and $e \approx 2.7$ is sufficient to obtain $\left\lfloor\frac{\pi e}{4}\right\rfloor = 2$.

16. B Note that the function is odd and the integral is equal to $\int_3^4 \frac{x^3}{\sqrt{16+x^2}}\,dx$. With a trigonometric substitution $x = 4\tan\theta$, $dx = 4\sec^2\theta\,d\theta$ and $\sqrt{16+x^2} = 4\sec\theta$ and the integral becomes $\int_{\tan^{-1}3/4}^{\pi/4}\frac{64\tan^3\theta}{4\sec\theta}\cdot 4\sec^2\theta\,d\theta = \int_{\tan^{-1}3/4}^{\pi/4}64\tan^3\theta\sec\theta\,d\theta = 64\int_{\tan^{-1}3/4}^{\pi/4}(\sec^2\theta - 1)\sec\theta\tan\theta\,d\theta$. $u = \sec\theta$ yields $64\int_{5/4}^{\sqrt{2}}(u^2-1)\,du = \frac{115-64\sqrt{2}}{3}$. $115 + 64 + 2 + 3 = 184$.

17. D $\lim_{x\to0^+}(\ln x\cdot\tan(2x)) = \lim_{x\to0^+}\left(\frac{\ln x}{\cot(2x)}\right) = \lim_{x\to0^+}\left(-\frac{1/x}{2\csc^2 x}\right) = \lim_{x\to0^+}\left(-\frac{\sin^2 x}{2x}\right) = -\frac{1}{2}\lim_{x\to0^+}\sin x = 0$.

18. B Rearranging, $(3y^2+1)\,dy = \left(3 + \frac{1}{x^2}\right)dx$. Integrating, $y^3 + y = 3x - \frac{1}{x} + C$. Setting $x = y = 1$ yields $C = 0$, so $f(x,y) = y^3 + y - 3x + \frac{1}{x}$. $f(k,0) = -3k + \frac{1}{k} = 0$, so $3k^2 = 1$ and $k = \frac{1}{\sqrt{3}}$.

19. E All four choices are classical rotated conic section invariants.

20. D Solving $2t^2 - 3t + 2 = 7$ gives $(2t-5)(t+1) = 0$. Solving $t^2 + 6t + 8 = 3$ gives $(t+5)(t+1) = 0$. Only $t = -1$ solves both of these equations. $\frac{dy}{dx} = \frac{2t+6}{4t-3}$.

By the Chain Rule, $\frac{d^2y}{dx^2} = \frac{d}{dt}\left[\frac{dy}{dx}\right]\frac{dt}{dx} = \frac{2(4t-3)-4(2t+6)}{(4t-3)^2} \cdot \frac{1}{4t-3} = -\frac{30}{(4t-3)^3}$. Setting $t = -1$ yields $\left.\frac{d^2y}{dx^2}\right|_{\langle x,y\rangle=\langle 3,7\rangle} = \frac{30}{343}$.

21. A $f'(x) = 3x^2 - 12x + 9$ and $f''(x) = 6x - 12$. $f(x)$ has an inflection point at $x = 2$. $f'(2) = 12 - 24 + 9 = -3$, so the slope of the normal line is $\frac{1}{3}$.

22. C Consider the cones as solids of rotation with their axes on the positive x-axis. The larger cone is the rotation of the line $y = \frac{20-x}{4}$ over the x-axis between $x = 0$ and $x = 20$. If the height of the smaller cone is x, then its radius is $\frac{20-x}{4}$ and its volume is $\frac{\pi}{48}x(20-x)^2$. The derivative of this is $\frac{\pi}{48}\left((20-x)^2 - 2x(20-x)\right) = \frac{\pi}{48}(3x^2 - 80x + 400) = \frac{\pi}{48}(x-20)(3x-20)$. $x \neq 20$, so let $x = \frac{20}{3}$. This gives a total volume of $\pi \cdot \frac{20 \cdot 40^2}{48 \cdot 27} = \frac{2^4 5^3}{3^4}$. The total number of factors of $2^4 3^4 5^3$ is $5 \cdot 5 \cdot 4 = 100$.

23. A Deriving again, $5f''(x) = 9f'(x) - 3g'(x) = \frac{9}{5}\left(9f(x) - 3g(x)\right) - \frac{3}{5}\left(2f(x) + 16g(x)\right)$, so $25f''(x) = 75f(x) - 75g(x)$ and $f''(x) = 3f(x) - 3g(x) = 5f'(x) - 6f(x)$. Setting $f''(x) - 5f'(x) + 6f'(x) = 0$ and factoring the characteristic equation yields $f(x) = c_1e^{2x} + c_2e^{3x}$, so the same must be true for $g(x)$: $g(x) = c_3e^{2x} + c_4e^{3x}$. Matching coefficients of e^{2x} and e^{3x} in the first equation yields $10c_1 = 9c_1 - 3c_3$ and $15c_2 = 9c_2 - 3c_4$, or $c_1 = -3c_3$ and $-2c_2 = c_4$. Combining these with the given $c_1 + c_2 = 1$ and $c_3 + c_4 = 3$ gives $\langle c_1, c_2, c_3, c_4\rangle = \langle 3, -2, -1, 4\rangle$, so $f(x) = 3e^{2x} - 2e^{3x}$ and $g(x) = -e^{2x} + 4e^{3x}$. Setting these equal to each other gives $4e^{2x} = 6e^{3x}$, or $e^x = \frac{2}{3}$.

24. C $4 + 8\cos\theta = 0$ at $\theta = \pi \pm \frac{\pi}{3}$, so the bounds of the area are $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$. Integrating, $\frac{1}{2}\int_{2\pi/3}^{4\pi/3}(4 + 8\cos\theta)^2\,d\theta = 8\int_{2\pi/3}^{4\pi/3}(1 + 4\cos\theta + 4\cos^2\theta)\,d\theta = 8\int_{2\pi/3}^{4\pi/3}(3 + 4\cos\theta + 2\cos(2\theta))\,d\theta = 3x + 4\sin\theta + \sin(2\theta)]_{2\pi/3}^{4\pi/3} = 8\left(2\pi - 3\sqrt{3}\right) = 16\pi - 24\sqrt{3}$.

25. C Using the Bounds Trick, $u = \frac{\pi}{2} - \theta$ yields $I = \int_0^{\pi/2}\frac{\cos^{2023}\theta}{\sin^{2023}\theta + \cos^{2023}\theta}\,d\theta$. Adding these together, $2I = \int_0^{\pi/2}\frac{\sin^{2023}\theta + \cos^{2023}\theta}{\sin^{2023}\theta + \cos^{2023}\theta}\,d\theta = \int_0^{\pi/2}d\theta = \frac{\pi}{2}$, so $I = \frac{\pi}{4}$ and $8092I = 2023\pi$.

26. B The Bounds Trick and Sum of Cubes factorization make this integral equal to $\frac{1}{2}\int_0^{\pi/2}\frac{d\theta}{\sin^2\theta - \sin\theta\cos\theta + \cos^2\theta} = \int_0^{\pi/2}\frac{d\theta}{2-\sin(2\theta)} = \frac{1}{2}\int_0^{\pi}\frac{d\theta}{2-\sin(\theta)}$. Using the Weierstrass substitution as suggested, this is $\int_0^1\frac{1+t^2}{2(1+t^2)-2t}\cdot\frac{2}{1+t^2}\,dt = \int_0^1\frac{dt}{t^2-t+1} = \int_0^1\frac{dt}{\left(t-\frac{1}{2}\right)^2+\frac{3}{4}} = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2t-1}{\sqrt{3}}\right)\Big]_0^1 = \frac{2\pi}{3\sqrt{3}}$.

27. A The sum is equal to $\sum_{n=1}^{\infty}\arctan\left(\frac{2}{n^2}\right)$. Since $\arctan u - \arctan v = \arctan\frac{u-v}{1+uv}$, $\arctan\frac{2}{n^2} = \arctan(n+1) - \arctan(n-1)$. The series telescopes, so its sum is

$2\arctan\infty - \arctan 0 - \arctan 1 = \pi - 0 - \frac{\pi}{4} = \frac{3\pi}{4}$. Using $\pi \approx 3.14$ is sufficient to approximate the value of the expression as 235, which has a sum of digits of 10.

28. B Let $u = x^2 + 1$, then $-x^2 = -u + 1$.

$$\int_1^\infty e^{-x^2}\, d(x^2+1) = \int_0^\infty e^{-u+1}\, du = e\int_0^\infty e^{-u}\, du = -e(e^{-u})\Big|_0^\infty = e$$

29. B Let α be the angle from the floor to the line between Anagh and the bottom of the screen, and let β be the angle from the floor to the line between Anagh and the top of the screen. We know $\tan\alpha = \frac{9}{x}$ and $\tan\beta = \frac{25}{x}$. The goal is to maximize $\arctan\frac{25}{x} - \arctan\frac{9}{x}$. The derivative of this is $\frac{9}{x^2+81} - \frac{25}{x^2+625}$. Setting this equal to 0 yields $9x^2 + 5625 = 25x^2 + 2025$ or $16x^2 = 3600$, or $x = 15$.

30. A $\frac{1}{2}\cdot\begin{vmatrix} -1 & -1 & 1 \\ 5 & 8 & 1 \\ 3 & 19 & 1 \end{vmatrix} = \frac{1}{2}(-8 - 3 + 95 - 24 + 19 + 5) = \frac{1}{2}\cdot 84 = 42$, so the answer to Life, The Universe, and Zeverything (like the answer to Life, The Universe, and Everything) is 42.