1. C

$$\int_1^2 \left(x+\frac{1}{x}\right)^3 dx \;=\; \int_1^2 x^3+3x+\frac{3}{x}+\frac{1}{x^3}\,dx$$

$$\int_1^2 x^3+3x+\frac{3}{x}+1/x^3 dx \;=\frac{69}{8}+\ln(8)$$

2. B

$$\int_0^1 x\sqrt{1-x}\,dx \to u=1-x \to \int_1^0 (u-1)\sqrt{u}\,du$$

$$\int_1^0 (u-1)\sqrt{u}\,du \;=\; \int_1^0 u\sqrt{u}-\sqrt{u}\,du \;=\; 4/15$$

3. A

$$\int_0^1 x\sqrt{1-x^4}dx \to u=x^2 \to \int_0^1 \frac{1}{2}\sqrt{1-u^2}du$$

If we let $y=\frac{1}{2}\sqrt{1-u^2}$ and then graph that in the uy plane, we will get a quarter of an ellipse with $a=1$ and $b=\frac{1}{2}$ so the area will be $\frac{\pi ab}{4}=\frac{\pi}{8}$

4. C

$$\int_0^4 x^3-6x^2+13x-7\,dx=\int_0^4 (x-2)^3+(x-2)+3\,dx$$

Since $(x-2)^3+(x-2)$ is symmetric about x=2, the midpoint Riemann sum will evaluate to 0 on this part. So, we only need to consider the midpoint Riemann sum on $\int_0^4 3\,dx$ which is going to be 12.

5. D

$$\int_0^4 |x^2-3x+2|\,dx=\int_0^1 |x^2-3x+2|\,dx+\int_1^2 |x^2-3x+2|\,dx+$$

$$\int_2^4 |x^2-3x+2|\,dx \to \int_0^4 |x^2-3x+2|\,dx=\frac{17}{3}$$

6. E

Either k is positive or negative. If k is positive: $\int_0^k kx-x^2\,dx=1$

If k is negative: $\int_k^0 kx-x^2\,dx=1$

$$\to\frac{k^3}{6}=\pm 1\to k=\pm\sqrt[3]{6}$$

7. E

$$\int_0^1 \frac{x^6}{\sqrt{1-x^2}}dx \to x=\sin u \to \int_0^{\frac{\pi}{2}} \sin^6(u)\,du$$

$\int_0^{\frac{\pi}{2}} \sin^a(u)\,du=\frac{\pi}{2}*\frac{a-1!!}{a!!}$ for even a, so

$$\int_0^{\frac{\pi}{2}} \sin^6(u)\,du=\frac{5\pi}{32}$$

8. A

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i^{n-1}}{n^n}=\int_0^1 x^{n-1}\,dx=\frac{1}{n}=0$$

9. A

$$\frac{\int_0^4 x^3+2x^2+3x+4}{4}=\frac{110}{3}$$

10. D

$$\int_0^\infty \frac{1}{x^n+1}\,dx = \frac{\frac{\pi}{n}}{\sin\frac{\pi}{n}}$$

$$\int_0^\infty \frac{1}{x^4+1}\,dx = \frac{\frac{\pi}{4}}{\sin\frac{\pi}{4}} = \frac{\pi\sqrt{2}}{4}$$

If you want, you can try the following:

$$\int_0^\infty \frac{1+x^2}{x^4+1}\,dx + \int_0^\infty \frac{1-x^2}{1+x^4}\,dx = \int_0^\infty \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx + \int_0^\infty \frac{\frac{1}{x^2}-1}{x^2+\frac{1}{x^2}}\,dx$$

From here, we have the following substitutions for the two integrals and the rest is up to you ☺

$$u = x - \frac{1}{x}$$
$$u = x + \frac{1}{x}$$

11. B

$$\int_1^2 \frac{4x^2+9x+2}{x^3+3x^2+x}\,dx = \int_1^2 \frac{4x^3+9x^2+2x}{x^4+3x^3+x^2}\,dx = \ln\frac{44}{5}$$

12. C Let's do complementary counting and find the probability it doesn't touch a line. Let's call the angle the needle makes with the horizontal θ, θ is going to be a random value from $0\ to\ 2\pi$. WLOG, let's assume θ is between $0\ and\ \frac{\pi}{2}$ In terms of theta, the vertical variation of the needle is $\sin\theta$ and the horizontal variation is $\cos\theta$, so now we consider where the center of the needle can be. It is a rectangle with sides $1 - \sin\theta$ and $1-\cos\theta$. So now we integrate:

$$\frac{\int_0^{\frac{\pi}{2}}(1-\sin\theta)(1-\cos\theta)d\theta}{\frac{\pi}{2}} = 1 - \frac{3}{\pi}$$, so our answer is $\frac{3}{\pi}$

13. A It is an odd function integrated from symmetric bounds, so it is 0

14. E

$$\int_0^x 6t - t^2 - 5\,dt$$

If we take the derivative, we get $6x - x^2 - 5x$. This has roots at 1 and 5, and the root at 5 is the relative maximum. However, plugging in endpoints: $-\infty\ gets\ \infty$, so the maximum doesn't exist.

15. A

$$\pi\int_0^1 (x^2+3x+5)^2\,dx = \frac{1441\pi}{30}$$

16. D

$$x^3 - 6x^2 + 12x - (x^2 - 2x + 8) = x^3 - 7x^2 + 14x - 8$$

This has roots at 1, 2, and 4, so we evaluate:

$$\int_1^2 |x^3 - 7x^2 + 14x - 8|\,dx + \int_2^4 |x^3 - 7x^2 + 14x - 8|\,dx = \frac{37}{12}$$

17. D

$$\int_0^4 \left(3t + \sqrt{t} - 1\right)dt = \frac{3}{2}t^2 + \frac{2}{3}t\sqrt{t} - t\,\Big|_0^4 = \frac{76}{3}$$

18. B

$$\int_0^\infty \frac{1}{x^6+x^4+x^2+1}dx = \int_0^\infty \frac{1}{(x^4+1)(x^2+1)}dx$$

$$u=\frac{1}{x} \to \int_\infty^0 \frac{1}{\left(\frac{1}{u^4}+1\right)\left(\frac{1}{u^2}+1\right)} \cdot -\frac{1}{u^2}du \to \int_0^\infty \frac{u^4}{(u^4+1)(u^2+1)}du$$

$$I = \int_0^\infty \frac{1}{(x^4+1)(x^2+1)}dx = \int_0^\infty \frac{u^4}{(u^4+1)(u^2+1)}du$$

$$2I = \int_0^\infty \frac{x^4+1}{(x^4+1)(x^2+1)}dx = \int_0^\infty \frac{1}{(x^2+1)}dx$$

$$2I = \arctan(\infty) - \arctan(0) = \frac{\pi}{2} \to I = \frac{\pi}{4}$$

19. C

$$I(a) = \int_{-\infty}^\infty e^{-ax^2}dx \to I'(a) = \int_{-\infty}^\infty -x^2e^{-ax^2}dx \text{ ...}$$

$$I'''(a) = \int_{-\infty}^\infty -x^6e^{-ax^2}dx$$

By doing $x=\frac{u}{\sqrt{a}}$ on $I(a)$ we get $I(a)=\frac{\sqrt{\pi}}{\sqrt{a}}$. Taking the derivative of this 3 times we get $I'''(a) = -\frac{15\sqrt{\pi}}{8a^{\frac{7}{2}}}$. Plugging 1 into this and into our other expression for $I'''(a)$ we get $I'''(1) = -\frac{15\sqrt{\pi}}{8} = \int_{-\infty}^\infty -x^6e^{-x^2}dx$, so $\int_{-\infty}^\infty x^6e^{-x^2}dx = \frac{15\sqrt{\pi}}{8}$

20. C

$$\lim_{a\to 0}\int_0^a \frac{a}{a^2+x^2}dx = \lim_{a\to 0}\frac{\pi}{4} = \frac{\pi}{4}$$

21. C Letting $u=\frac{x}{2}$, we can turn the integral into $\int_0^{\frac{\pi}{3}} 2\sec(u)\tan(u)\,du = 2\sec(u)\Big|_0^{\frac{\pi}{3}} = 2$

22. C

$$\int_0^\infty e^{-x}(\lfloor x\rfloor)^2dx = \int_1^2 e^{-x}dx + \int_2^3 4e^{-x}dx \text{ ...}$$

$$\int_0^\infty e^{-x}(\lfloor x\rfloor)^2dx = \frac{1}{e}-\frac{1}{e^2}+\frac{4}{e^2}-\frac{4}{e^3}\text{ ...} = \frac{1}{e}+\frac{3}{e^2}+\frac{5}{e^3}\text{ ...}$$

$$\frac{1}{e}+\frac{3}{e^2}+\frac{5}{e^3}\text{ ...} = \left(\frac{1}{e}+\frac{1}{e^2}\text{ ...}\right)+\left(\frac{2}{e^2}+\frac{4}{e^3}\text{ ...}\right) = \frac{\frac{1}{e}}{1-\frac{1}{e}} + \frac{\frac{2}{e^2}}{\left(1-\frac{1}{e}\right)^2} = \frac{e+1}{(e-1)^2}$$

23. A We let $u=1+7x^2$ and then $\frac{1}{14}\int_1^8 \frac{1}{u^2}du = \frac{1}{16}$

24. E

$$\lim_{n\to\infty}\sum_{i=0}^{n^2}\frac{i}{n^2}$$, since the upper bound is n^2 not n, our sum will diverge or DNE.

25. C We let $u=\frac{x}{8}$ and so we get $\int_0^{\frac{\pi}{4}} 8\sec^2(u)\,du = 8$

26. C Using power rule on the left-hand side, we get $\frac{4a}{3}+\frac{1}{2} = 3a \to a = .3$

27. A Counterexamples for all 3 are as follows:

I: $\cos x^2$

II: $\cos x^2$

III: $\cos x^2$

28. B $\int_0^\infty \frac{1}{(1+x)(1+(\ln x)^2)}dx \to x = e^u \to \int_{-\infty}^\infty \frac{1}{(1+e^u)(1+u^2)}du \to u = 1 - v \to$

$$I = \int_{-\infty}^\infty \frac{1}{(1+e^{-v})(1+v^2)}dv = \int_{-\infty}^\infty \frac{e^u}{(1+e^u)(1+u^2)}du$$

$$= \int_{-\infty}^\infty \frac{1}{(1+e^u)(1+u^2)}du$$

$$2I = \int_{-\infty}^\infty \frac{(1+e^u)}{(1+e^u)(1+u^2)}du = \int_{-\infty}^\infty \frac{1}{(1+u^2)}du = \pi \to I = \frac{\pi}{2}$$

29. A $\int_0^{\frac{\pi}{4}} 2\tan x\,(\sin x)^2\, e^{(\tan x)^2}dx \to x = \arctan(u) \to \int_0^1 2u\frac{u^2}{u^2+1}e^{u^2}\frac{1}{u^2+1}du$

$$v = u^2 \to \int_0^1 \frac{v}{v+1}e^v\frac{1}{v+1}dv = \frac{e^v}{v+1} \to \frac{e}{2} - 1 \to \frac{e-2}{2}$$

30. E $\lim_{x\to 0}\sqrt{\sin x}$ doesn't exist since the limit from the left is nonexistent