All uppercase letter variables are positive integers unless otherwise stated. All fractions containing uppercase letter variables are in lowest terms. NOTA means "None of the Above."

~~~~~~ *Good luck, and have fun!* ~~~~~~

1. An Egyptian fraction expansion of a rational number q is a sequence of distinct positive integers $a_1 < a_2 < \cdots < a_n$ such that $\sum_{i=1}^{n} \frac{1}{a_i} = q$. The greedy algorithm for finding an Egyptian fraction of a number involves the recursive addition of the largest unit fractions possible until the desired result is achieved. For example, $\frac{7}{15} = \frac{1}{3} + \frac{2}{15} = \frac{1}{3} + \frac{1}{8} + \frac{1}{120}$. This can be expressed mathematically as $\frac{x}{y} = \frac{1}{\left\lceil \frac{y}{x} \right\rceil} + \frac{(-y) \bmod x}{y\left\lceil \frac{y}{x} \right\rceil}$.
When this algorithm is applied to the number $\frac{4}{17}$, $a_4 = 3{,}039{,}345$. Find a_3.
A. 765 B. 1233 C. 1305 D. 1479 E. NOTA

2. Find the sum of an infinite geometric series whose common ratio is $-\frac{1}{3}$ and whose third term is 80.
A. 120 B. 540 C. 1020 D. 1080 E. NOTA

3. If $0.13\overline{37} = 0.13373737\ldots = \frac{A}{B}$, find $A + B$.
A. 2551 B. 2806 C. 3151 D. 5663 E. NOTA

4. If $-\sum_{n=0}^{15} n \operatorname{cis} \frac{n\pi}{4} = A + B\sqrt{2} + Ci + Di\sqrt{2}$, find $A + B + C + D$.
A. 0 B. 16 C. 24 D. 32 E. NOTA

5. Find $\left(\sin \frac{\pi}{1024}\right) \prod_{n=3}^{10} \cos \frac{\pi}{2^n}$.
A. $\frac{1}{1024}$ B. $\frac{1}{512\sqrt{2}}$ C. $\frac{1}{512}$ D. $\frac{1}{256\sqrt{2}}$ E. NOTA
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6. Luke drops a big bouncy ball from a 50-foot tall building. Every time the ball hits the ground, it rebounds to $\frac{4}{5}$ of the height it reached on the previous bounce. As the number of bounces approaches infinity, find the total vertical distance the ball traveled in feet.

A. 250 B. 400 C. 450 D. 500 E. NOTA

7. The 28^{th} term of an arithmetic series is 42, and the 85^{th} term is 111. Find the 2023^{rd} term.

A. 2112 B. 2369 C. 2403 D. 2457 E. NOTA

8. For a cubic function $f(x)$, the values of $f(1)$, $f(2)$, $f(3)$, and $f(4)$ are 1, 2, 3, and 4 in some order. Find the maximum possible value of $f(5)$.

A. 15 B. 17 C. 20 D. 21 E. NOTA

For questions 9 and 10, use the following information. Anagh the Ant is standing at the origin. He travels to the point $(5,5)$, moving in increments of 1 unit either directly up or to the right (he does not travel any other direction). Every movement begins at a lattice point.

9. Find the number of unique paths Anagh can take that do not pass through the point $(3,4)$.

A. 147 B. 168 C. 189 D. 214 E. NOTA

10. Find the number of unique paths Anagh can take that do not cross the line $y = x$.
Note: The path can touch the line, but it cannot pass *through* it.

A. 42 B. 45 C. 50 D. 54 E. NOTA

11. Find the radius of convergence of the sum $\sum_{n=1}^{\infty} \frac{(n+1)^{2023} x^n}{n^n}$.

A. 0 B. 1 C. 2023 D. ∞ E. NOTA

12. Determine the convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^n 4n}{4n^4+1}$.

A. Absolutely convergent B. Conditionally convergent
C. Divergent D. Impossible to determine E. NOTA

13. A recursive sequence $\{a_n\}_{n\geq 0}$ is defined by $a_0 = 9$, $a_1 = 1$, $a_2 = 19$, and for all $n \geq 3$, $a_n = 6a_{n-1} - 5a_{n-2} - 12a_{n-3}$. If $a_n = rx^n + sy^n + tz^n$ for integers r, s, t, x, y, z, find $r + s + t + x + y + z$.

A. 15 B. 17 C. 19 D. 20 E. NOTA

14. Find the value of $k \in [250,500]$ that minimizes $|\prod_{n=0}^{10}(n^3 - k + n)|$.

A. 300 B. 324 C. 350 D. 496 E. NOTA

15. If $\sum_{i=0}^{10} x_i = 9$ and $\sum_{i=0}^{10}(x_i - 1)^2 = 99$, find $\sum_{i=0}^{10} x_i^2$.

A. 97 B. 98 C. 105 D. 106 E. NOTA

16. Let $f(x) = 2x^3 + 4x - 5$. Let L, R, T, and S respectively be the left-hand Riemann sum, right-hand Riemann sum, trapezoidal sum, and Simpson's approximations for $\int_1^3 f(x)\,dx$ with 4 subintervals. Find $3R - L - 2T + S$.

A. 104 B. 105 C. 106 D. 108 E. NOTA

17. Evaluate: $\sum_{n=1}^{\infty}\left((\sum_{k=1}^{n} k)(\sum_{k=1}^{n} k^3)^{-1}\right)$.

A. $\frac{3}{2}$ B. $\frac{\pi^2}{6}$ C. $\frac{7}{4}$ D. 2 E. NOTA

18. Evaluate: $\sum_{n=0}^{\infty}\left(\left(\frac{\pi}{3}\right)^n - \left(\frac{\pi}{4}\right)^n\right)$

A. $\frac{3}{\pi-3}$ B. $\frac{\pi}{\pi-3}$ C. $\frac{\pi}{\pi^2-7\pi+12}$ D. $\frac{24-7\pi}{\pi^2-7\pi+12}$ E. NOTA

19. Determine the convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^n \cos(\pi n)}{n}$.

A. Absolutely convergent B. Conditionally convergent
C. Divergent D. Impossible to determine E. NOTA

20. The graph of a quartic function $f(x)$ contains the points $(1,1)$, $(2,4)$, $(3,9)$, $(4,16)$, and $(5, n)$. Find the number of possible $0 \leq n \leq 625$ such that $f(x)$ has integer coefficients.

A. 24 B. 25 C. 26 D. 27 E. NOTA

21. Evaluate: $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{n^2+k^2}$.

A. 1 B. 2 C. $\frac{\pi}{4}$ D. $\frac{\pi}{2}$ E. NOTA

22. A sequence $\{a_n\}_{n\geq 0}$ of real numbers with initial values $-a_0 = a_1 = \sqrt{3}$ and $a_n > 0$ for all $n \geq 2$ satisfies the recurrence relation $a_{n+2} = \frac{a_n}{4}(1 - a_{n+1}^2)(1 - a_{n+2}^2)$ for all $n \geq 0$. Find $\lim_{n\to\infty} 3 \cdot 2^n a_n$.

A. $\sqrt{3}$ B. $3\sqrt{3}$ C. π D. 2π E. NOTA

23. Find the number of subsets of $\{1,2,3, \ldots, 10\}$ that contain no more than one out of any three consecutive integers. For example, $\{1,4,8\}$ is a valid set, but $\{1,3,8\}$ is not because it contains both 1 and 3 from the set $\{1,2,3\}$.

A. 60 B. 96 C. 105 D. 144 E. NOTA

24. For a sequence $\{a_n\}_{n\geq 0}$, let $\rho_n = n\left(\frac{a_n}{a_{n+1}} - 1\right)$. *Raabe's test* states that if $\lim_{n\to\infty} \rho_n > 1$, then $\sum_{n=0}^{\infty} a_n$ converges, and if $\lim_{n\to\infty} \rho_n < 1$, then $\sum_{n=0}^{\infty} a_n$ diverges. Determine the convergence of $\sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!} \cdot \frac{1}{2n+1}$, where $a!!$ is the double factorial, $a!! = a(a-2)(a-4)$ ....

A. Absolutely convergent B. Conditionally convergent
C. Divergent D. Impossible to determine E. NOTA

25. Find the sum of the solutions to $x - 2x^2 + x^3 - 2x^4 + x^5 - 2x^6 + \cdots = -\frac{2}{5}$.

A. $\frac{5}{12}$ B. $\frac{1}{2}$ C. $\frac{2}{3}$ D. $\frac{11}{12}$ E. NOTA

26. Find the coefficient of the $(x-2)^4$ term in the Taylor series expansion of $\frac{1}{x^2}$ centered about $x = 2$.

A. $\frac{3}{64}$ B. $\frac{5}{64}$ C. $\frac{3}{32}$ D. $\frac{5}{32}$ E. NOTA

27. $\{a_n\}_{n\geq 0}$ is an increasing arithmetic sequence and $\{b_n\}_{n\geq 0}$ is a geometric sequence. If $b_0 = 1$, $a_1 = b_1$, $a_2 = b_2$, and $3a_3 = b_3$, then $-a_0 = A + B\sqrt{C}$ where C is squarefree. Find $A + B + C$.

A. 10 B. 13 C. 19 D. 23 E. NOTA

28. Two distinct, real, infinite geometric series each have a sum of 1 and a common second term. The third term of one of the series (whose terms are all positive) is $\frac{1}{8}$. Find the third term of the other series.

A. $\frac{\sqrt{5}-2}{8}$ B. $\frac{\sqrt{5}-2}{4}$ C. $\frac{\sqrt{5}-1}{8}$ D. $\frac{\sqrt{5}-1}{4}$ E. NOTA

29. Determine the convergence of the series $\sum_{n=1}^{\infty} \frac{\sin n}{n+2\cos n}$.

A. Absolutely convergent B. Conditionally convergent
C. Divergent D. Impossible to determine E. NOTA

30. Rayo's number is an extraordinarily large positive integer, far larger than Graham's number and TREE[3], that was invented in a "large number battle" (a competition to see who could define the largest number) by Agustin Rayo in 2007. It is defined as the smallest positive integer bigger than any finite positive integer named by an expression in the language of first-order set theory with a googol (10^{100}) symbols or less. According to the Googology wiki, it is the largest "useful" number that has been used in a proof and is thus the largest uncontested "googologism".

Let R equal Rayo's number. Find $\sum_{r=R}^{\infty} \frac{1}{r}$.

A. 0 B. $\frac{1}{R-1}$ C. $\frac{R}{R^2-1}$ D. $\frac{R}{R^2-R+1}$ E. NOTA