BBBCD CDCAE DAACD CDECC CDAAA BCABE

1. B $\frac{4}{17}=\frac{1}{5}+\frac{3}{85}=\frac{1}{5}+\frac{1}{29}+\frac{2}{2465}=\frac{1}{5}+\frac{1}{29}+\frac{1}{1233}+\frac{1}{3039345}$. $a_3=1233$.
2. B The second term is -240 and the first term is 720. The sum of the geometric series as a whole is $\frac{720}{1+\frac{1}{3}}=540$.
3. B $0.13\overline{37}=0.13+0.00\overline{37}=\frac{13}{100}+\frac{37}{9900}=\frac{1324}{9900}=\frac{331}{2475}$. $331+2475=2806$.
4. C Pair terms up with terms 4 ahead in their sequence. Matching $n=0$ with $n=4$ and continuing through $n=3$ and $n=7$ yields a sum of $-4\left(1+\frac{1+i}{\sqrt{2}}+i+\frac{-1+i}{\sqrt{2}}\right)=-4\left(1+i+i\sqrt{2}\right)$. This pattern would repeat in $n=8$ through $n=15$, so the total sum is $-8-8i-8i\sqrt{2}$. $A+B+C+D=24$.
5. D Use of the double angle formula repeatedly yields $\frac{1}{2^8}\sin\frac{\pi}{4}=\frac{1}{256\sqrt{2}}$.
6. C The distance the ball goes down is $\frac{50}{1-\frac{4}{5}}=250$. The distance that the ball goes up is $\frac{50\cdot\frac{4}{5}}{1-\frac{4}{5}}=200$. The total distance is 450.
7. D The sequence increases by 69 every 57 terms, or 23 every 19 terms. $2023=85+1938=85+102\cdot 19$. The 2023^{rd} term is $111+102\cdot 23=2457$.
8. C By the Method of Finite Differences, we obtain the following.

a	b	c	d
$-a+b$	$-b+c$	$-c+d$	$-a+4b-6c+3a$
$a-2b+c$	$b-2c+d$	$-a+4b-5c+2d$	
$-a+3b-3c+d$	$-a+3b-3c+d$		

   Thus, $f(5)=-a+4b-6c+4d$. Setting $a=2$, $b=3$, $c=1$, and $d=4$ gives $f(5)=20$.
9. A The number of ways to go from the origin to $(5,5)$ is $\binom{10}{5}=252$. The number of ways to go from the origin to $(3,4)$ is $\binom{7}{4}=35$. The number of ways to go from here to $(5,5)$ is $\binom{3}{2}=3$, making for a total of 105 paths that go through $(3,4)$ and 147 that do not.
10. E The number of paths that do not go *above* the line $y=x$ is well-known to be the Catalan numbers $C_n=\frac{\binom{2n}{n}}{2n+1}$. $C_5=\frac{252}{6}=42$. The number of paths that do not cross the line is twice this, or 84.
11. D Applying the Root Test to this sequence shows that the limit goes to 0 for all x, and the radius of convergence is ∞.
12. A This sequence is convergent by the Alternating Series Test. Ignoring the $(-1)^n$, this series can be compared with $\sum\frac{1}{n^3}$ and is absolutely convergent.
13. A This is a homogenous recurrence relation, so it can be solved with the equation $x^3=6x^2-5x-12$. This has roots -1, 3, and 4, so $a_n=r(-1)^n+s\cdot 3^n+t\cdot 4^n$. Solving this with the initial conditions yields $a_n=6\cdot(-1)^n+5\cdot 3^n-2\cdot 4^n$. $6-1+5+3-2+4=15$.
14. C Note that when $n=7$, $n^3=343$, so setting $k-n=343$ would give a product of 0. This yields $k=350$.

15. D $\sum_{i=0}^{10}(x_i-1)^2=\sum_{i=0}^{10}x^2-2\sum_{i=0}^{10}x+\sum_{i=0}^{10}1=\sum_{i=0}^{10}x^2-18+11=99$, so $\sum_{i=0}^{10}x^2=106$.

16. C Note that $3R-L-2T+S=2(R-L)+(R+L-2T)+S$. The middle term always equals 0. The third term is the value of the integral. The first term is the difference between the right and left endpoints multiplied by the interval size. $2\cdot\frac{f(3)-f(1)}{2}=61-1=60$. $\int_1^3(2x^3+4x-5)\,dx=\frac{x^4}{2}+2x^2-5x\Big]_1^3=\frac{87}{2}+\frac{5}{2}=46$. The sum of these is 106.

17. D The summand is equal to $\frac{n(n+1)/2}{n^2(n+1)^2/4}=\frac{2}{n(n+1)}$. The sum of this is a telescoping series, because $\frac{2}{n(n+1)}=\frac{2}{n}-\frac{2}{n+1}$. Thus, $\sum_{n=1}^{\infty}\left(\frac{2}{n}-\frac{2}{n+1}\right)=\frac{2}{1}=2$.

18. E $\pi>3$ and the series diverges.

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19. C When n is an integer, $\cos\pi n=(-1)^n$ and the sum is $\sum_{n=1}^{\infty}\frac{1}{n}$, which is divergent.

20. C Consider $g(x)=f(x)-x^2$. This is also quartic and has roots at 1, 2, 3, and 4, so $g(x)=k(x-1)(x-2)(x-3)(x-4)$ and $f(x)=k(x-1)(x-2)(x-3)(x-4)+x^2$. $f(5)=24k+25$. The number of integers between 0 and 625 that are equivalent to 1 mod 24 is 27. However, $k=0$ would result in a quadratic, not a quartic function, so the number of possible k is 26.

21. C Multiplying by $\frac{n^2}{n^2}$, the sum is equal to $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1/n}{1+k^2/n^2}$. This is a Riemann sum that can be converted to the integral $\int_0^1\frac{dx}{1+x^2}=\arctan 1-\arctan 0=\frac{\pi}{4}$.

22. D Noting that $a_2=\frac{\sqrt{3}}{3}$, rearrange to set $a_n=\frac{4a_{n+2}}{(1-a_{n+2}^2)(1-a_{n+1}^2)}=\frac{2a_{n+2}}{1-a_{n+2}^2}\cdot\frac{2}{1-a_{n-1}^2}$ and recognize the first double-angle formula for tangent, which would multiply by the second term to form the double-angle formula again. If $a_{n+2}=\tan\theta_{n+2}$, then $a_{n+1}=\tan 2\theta_{n+2}$ and $a_n=\tan 4\theta_{n+2}$. Thus, $a_n=\tan\frac{\theta_0}{2^n}$. The only angle that satisfies $a_n>0$ for all $n\geq 2$ is $\theta_0=\frac{2\pi}{3}$. Thus, $a_n=\tan\frac{2\pi}{3\cdot 2^n}$ and $\lim_{n\to\infty}3\cdot 2^n\tan\frac{2\pi}{3\cdot 2^n}=\lim_{u\to 0}\frac{\tan 2\pi u}{u}=2\pi$.

23. A Consider the set $\{1,2,\dots,n\}$. For $n=1$, there are 2 valid subsets, and for $n=2$, there are 3 valid subsets. For larger values of n, the value n would have to be added to a subset present for the $n-3$ case. The recursion $a_n=a_{n-1}+a_{n-3}$ can be employed to find $a_{10}=60$.

24. A $\frac{a_n}{a_{n+1}}-1=\left[\frac{(2n-1)!!}{(2n)!!}\cdot\frac{1}{2n+1}\right]\Big/\left[\frac{(2n+1)!!}{(2n+2)!!}\cdot\frac{1}{2n+3}\right]-1=\frac{2n+2}{2n+1}\cdot\frac{2n+3}{2n+1}-1=\frac{4n^2+10n+6}{4n^2+4n+1}-1=\frac{6n+5}{4n^2+4n+1}$. $\lim_{n\to\infty}\frac{6n^2+5n}{4n^2+4n+1}=\frac{3}{2}>1$, so the sum is absolutely convergent.

25. A The sum is equal to $\frac{x}{1-x^2}-\frac{2x^2}{1-x^2}=\frac{x-2x^2}{1-x^2}$. Setting this equal to $-\frac{2}{5}$ yields $10x^2-5x=2-2x^2$, or $12x^2-5x-2=0$. This factors to $(4x+1)(3x-2)=0$, so $x=\frac{2}{3}$ or $x=-\frac{1}{4}$. These sum to $\frac{5}{12}$.

26. B The fourth derivative of $\frac{1}{x^2}$ at $x=2$ is $\frac{15}{8}$. Dividing by $4!=24$ gives the coefficient of the $(x-2)^4$ term, $\frac{5}{64}$.

27. C Let the common ratio of $\{b_n\}$ be r. If $b_0 = 1$, then $a_1 = b_1 = r$ and $a_2 = b_2 = r^2$ for some $r > 1$. $\{a_n\}$ has a common difference of $r^2 - r$, so $a_3 = 2r^2 - r$. Thus, $6r^2 - 3r = r^3$, so $r^2 - 6r + 3 = 0$ and $r = 3 + \sqrt{6}$. $a_0 = 2r - r^2 = 2(3+\sqrt{6}) - (3+\sqrt{6})^2 = (6 + 2\sqrt{6}) - (15 + 6\sqrt{6}) = -(9 + 4\sqrt{6})$. $9 + 4 + 6 = 19$.

28. A Let the second term of the mentioned series be x. Then its common ratio is $\frac{1}{8x}$ and its first term is $8x^2$. The sum of this series is $\frac{8x^2}{1-\frac{1}{8x}} = 1$. $64x^3 - 8x + 1 = 0$. This factors to $(4x-1)(16x^2+4x-1) = 0$, so $x = \frac{1}{4}$ or $x = \frac{-1\pm\sqrt{5}}{8}$. Investigating $x = \frac{1}{4}$ results in two identical series. $x = \frac{-1+\sqrt{5}}{8}$ is the other positive second term. The first term of this series is $\frac{3-\sqrt{5}}{4}$. Let the first term of the other series be y. Then the common ratio is $\frac{-1+\sqrt{5}}{8y}$ and the sum is $\frac{y}{1-\frac{-1+\sqrt{5}}{8y}} = 1$. This simplifies to $8y^2 - 8y + \sqrt{5} - 1 = 0$. The solutions to this are $\frac{3-\sqrt{5}}{4}$ and $\frac{1+\sqrt{5}}{4}$. The first of these would result in identical series, so $y = \frac{1+\sqrt{5}}{4}$. This results in a common ratio of $\frac{3-\sqrt{5}}{4}$ and a third term of $\frac{\sqrt{5}-2}{8}$.

29. B $\frac{\sin n}{n+2\cos n} = \frac{\sin n}{n} + \frac{\sin n}{n+2\cos n} - \frac{\sin n}{n} = \frac{\sin n}{n} - \frac{2\cos n \sin n}{n(n+2\cos n)}$. Comparison to $\frac{\pm 2}{n(n-2)}$ shows that the second term is absolutely convergent. $\sum \frac{\sin n}{n}$ can be shown to be conditionally convergent by Dirichlet’s test.

30. E Every infinite contiguous subsequence of the harmonic series is divergent.