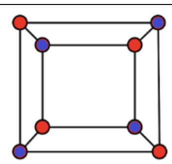
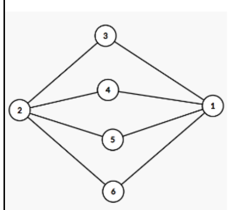
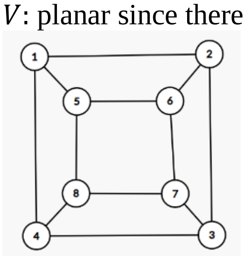
1	B
2	C
3	C
4	C
5	C
6	D
7	D
8	C
9	A
10	B
11	E
12	C
13	B
14	A
15	C
16	D
17	A
18	E
19	C
20	E
21	B
22	D
23	B
24	A
25	B
26	C
27	A
28	D
29	B
30	E

#	Ans	Solution
1	B	In Boolean algebra, we have $$1+1\times 1+0=1+1+0=1+0=1.$$
2	C	$S=\{1,2,3\}\cup$ subset of $\{4,5,6,7\}$. There are $2^4=16$ different possibilities for the subsets – empty set as one of the inequalities is strict.
3	C	$99^{17}=(100-1)^{17}$. Only the last two terms of this binomial distribution matters when taken mod 1000 since $100^2=0$ (mod 1000). $(100-1)^{17}\equiv 1700-1\equiv 699$ (mod 1000).
4	C	$I: S\in\mathbb{N}$ $II: S=\mathbb{N}\times\mathbb{N}=\mathbb{N}$ $III: S=\mathbb{Q}\times\mathbb{Q}=\mathbb{N}$ IV: it is well known that $\mathbb{R}^+$ is not countable. You can make a bijection from $(0,1)$ to $\mathbb{R}^+$ by mapping x to $\frac{1}{x}-1$. x Thus, reals from 0 to 1 is not countable. V: the unit circle can be mapped to reals from 0 to 2π. Since $\|V\|\geq\|IV\|$, V is also not countable.
5	C	From the description $A\cup B=\mathbb{N}$. I: $f(x)=1, g(x)=0$ is a counterexample. II: $f(x)=0, g(x)=0$ is a counterexample. III: Since the union is an infinite set, at least 1 of A, B must be an infinite set. IV: Using III, B must be an infinite set. V: $f(x)=0, g(x)=0$ is a counterexample.
6	D	The expression can be rewritten as $a+(a+b)+(a+b+c)=30$. Since $a,b,c>0$, $a<a+b<a+b+c$. Thus the number of solutions is equal to the number of increasing positive integer triplets (x,y,z) that satisfy $x+y+z=30, x<y<z$. We proceed with complimentary counting. There are $\binom{29}{2}=406$ triplets that satisfy x+$y+z=30$. There are $3\cdot 13$ triplets where exactly two numbers are equal. There is 1 triplet where all 3 numbers are equal. Finally divide by 3! since there are 3! ways to order (a,b,c). $\frac{406-39-1}{3!}=61$. *Alternatively, case work on a, then b will also do the job.
7	D	We have a simple directed graph, so we can generate the adjacency matrix A with $$A_{i,j}=\begin{cases}1 \textit{ if there is a directed edge from i to j,}\\ 0 \textit{ otherwise.}\end{cases}$$ Doing this, we get answer choice D.
8	C	First, it is easy to see there is a Hamiltonian Circuit. (Going around in a circle.) There are two vertices with odd degree, thus there is a Euler Path, but not an Euler Circuit.
9	A	I: There is no multiplicative inverse. II: Satisfies all the conditions. III: Is not closed under addition. IV: Satisfies all the conditions. V: There is no additive inverse.
10	B	The diameter is the longest distance between two vertices. It isn't too hard to see that the longest distance is from 1 to 3.
11	E	Note that $b\to c\Leftrightarrow\neg b\vee c$. Now, repeatedly using De Morgan's laws, we get

		$\Leftrightarrow \neg((a \wedge (\neg b \vee c)) \vee \neg d),$ $\Leftrightarrow \neg\big(a \wedge (\neg b \vee c)\big) \wedge d,$ $\Leftrightarrow \big(\neg a \vee \neg(\neg b \vee c)\big) \wedge d,$ $\Leftrightarrow \big(\neg a \vee (b \wedge \neg c)\big) \wedge d.$ Note that this is not equivalent to answer choice B because of order of logical operations.
12	C	Using the conditions, $f(5) = 1$. We proceed with case work on $f(4)$. Case 1: $f(4) = 1$. $f(1), f(2), f(3)$ must contain 2,3. Using PIE, there are $3^3 - 2^3 - 2^3 + 1 = 12$ ways to do this. Case 2: $f(4) = 2$. $f(1), f(2), f(3)$ must contain 3. Using PIE, there are $3^3 - 2^3 = 19$ ways to do this.
13	B	In base 10, the sum of digits preserves modulo 9 since $10^n = 1 \pmod 9$. Similarly, in base 5, the sum of digits preserves modulo 4. Since $n^2 \equiv n \pmod{4,9} \rightarrow n(n-1) = 0 \rightarrow n = 0,1 \pmod{4,9}$. There are $2 \cdot 2 = 4$ possibilities mod 36, namely 0,1,9,28. Checking numbers starting from 28, the smallest solution is 45. ($45^2 = 2025, 140_5 = 31100_5$)
14	A	A full binary tree is a binary tree where each node has exactly 0 or 2 children. The maximum height is when each layer has one node with two edges. Thus, a full binary tree with h levels will have $2h - 1$ nodes. $M = 501$. The minimum number of levels is when everything is as full as possible. If the tree has 10 levels and is completely full, it contains 1023 nodes. For a tree with 1001 nodes, simply remove 11 pairs of nodes from the bottom of the tree. $m = 10$. Thus $M - m = 501 - 10 = 491$.
15	C	A: 2  B: Having one group of vertices blue, and the other group red means that the chromatic number is 2. C: Since each vertex is connected to 2 other vertices, the chromatic number is 3. D: Alternating between red and blue gives a chromatic number of 2.
16	D	There will be $d(n)$ (number of factors of n) numbers such that $\frac{n}{k}$ is an integer (since all factors are less than or equal to n), and $\left\lfloor \frac{49}{n} \right\rfloor$ numbers such that $\frac{k}{n}$ is an integer, with $k = n$ being double-counted. Thus, we have $$d(n) + \left\lfloor \frac{49}{n} \right\rfloor - 1 = 4 \rightarrow d(n) + \left\lfloor \frac{49}{n} \right\rfloor = 5.$$ We proceed with case work on $\left\lfloor \frac{49}{n} \right\rfloor$, noting that $d(n) \geq 2$ for all $n > 1$. Case 1: $\left\lfloor \frac{49}{n} \right\rfloor = 1 \rightarrow 25 \leq n \leq 49$. Thus, $d(n) = 4 \rightarrow n = pq$ or p^3 for primes p, q. Checking, $n = 26, 27, 33, 34, 35, 38, 39, 46$ Case 2: $\left\lfloor \frac{49}{n} \right\rfloor = 2 \rightarrow 17 \leq n \leq 24$. Thus, $d(n) = 3 \rightarrow n = p^2$ for prime p. There are no possible values of n.

		Case 3: $\left\lfloor\frac{49}{n}\right\rfloor = 3 \to 13 \le n \le 16$. Thus, $d(n) = 2 \to n =$ prime. The possible values are 13. Our total is $8 + 1 = 9$.
17	A	I, II: Since $S_a \subseteq S_b$ if $a \le b$, both I, II are true. III, IV: $\|S_a\| = (2a+1)(a+1) = 2a^2 + 3a + 1 \to \|S_b\| - \|S_a\| = 2b^2 - 2a^2 + 3b - 3a = (b-a)(2a+2b+3)$ III: If $b = a+1, \|S_b\| - \|S_a\| = 2(a+b) + 3 < 3(a+b)$ if a, b are large. IV: $b = 2, a = 1$ is a counterexample.
18	E	As k grows to infinity, x will be all numbers in the form $2^a 3^b$. The sum of $\frac{1}{x^2}$ is $\left(1 + \frac{1}{4} + \frac{1}{16} \dots\right)\left(1 + \frac{1}{9} + \frac{1}{81} \dots\right) = \frac{1}{1-\frac{1}{4}} \cdot \frac{1}{1-\frac{1}{9}} = \frac{4}{3} \cdot \frac{9}{8} = \frac{3}{2}$. However, we need to subtract the elements in S_1 since it is the intersection between S_k and the complement of S_1. $\frac{3}{2} - \left(1 + \frac{1}{4} + \frac{1}{16}\right)\left(1 + \frac{1}{9}\right) = \frac{3}{2} - \frac{21}{16} \cdot \frac{10}{9} = \frac{1}{24}$.
19	C	It is easy to find an example where $K_{3,4}$ has 2 intersections. Thus, the crossing number is less than or equal to 2. Now we aim to prove that the crossing number is indeed 2. For the sake of contradiction, let the crossing number be 1. (It cannot be 0 since it contains $K_{3,3}$) In the crossing, there will be 4 vertices involved: 2 from the group of 4, and 2 from the group of 3. If we remove one of the vertices from the group of 4 along with all edges connected, the crossing number will now be 0 since there are no more intersections. However, since the crossing number of $K_{3,3} = 1$, this is a contradiction.
20	E	I: well known to be non-planar II: planar since there are no intersections as shown below  III: since it contains K_5, it is non-planar IV: planar since there are no intersections V: planar since there are no intersections as shown below 
21	B	$2025^2 = 45^4 = 3^8 \cdot 5^4$. First, we aim to prove that x, y are both a multiple of 3. It is easy to see that if x is not a multiple of 3, $x^2 = 1$ (mod 3). Then if at least one of x or y isn't a multiple of 3, we get $$x^2 + y^2 \equiv 1,2 \ (mod\ 3),$$ which is a contradiction.

		This tells us that x, y are both multiples of 81. Now we need to find the number of lattice points with $x^2 + y^2 = 5^4 = 625$. This is just the Pythagorean triples with hypotenuse 25, which are $(7,24,25), (15,20,25)$. These produce 8 solutions each $(\pm x, \pm y), (\pm y, \pm x)$, and the trivial solution (0,25) produces 4. The total is 20.
22	D	First, we calculate the probabilities of each of the scenarios. 3-people game: Draw: 3 (everyone has the same hand) + 6 (everyone has a different hand) = $\frac{3+6}{27} = \frac{1}{3}$. 2 people win/1 person wins: By symmetry, both are $\frac{1}{3}$. 2-people game: Draw: $\frac{1}{3}$, Win: $\frac{2}{3}$ Let E_3 be the expected number of games until a single winner arises in a 3-person game, and let E_2 be the expected number of games until a single winner arises in a 2-person game. $$E_2 = E_2 \cdot \frac{1}{3} + 1 \rightarrow E_2 = \frac{3}{2}.$$ $$E_3 = E_3 \cdot \frac{1}{3} + E_2 \cdot \frac{1}{3} + 1 \rightarrow \frac{2E_3}{3} = \frac{3}{2} \rightarrow E_3 = \frac{9}{4}.$$
23	B	Consider a 5 by 4 grid where x is the number of heads, and y is the number of tails. This is a typical problem where you go up the grid and add up the square from the left and from below; however, you cannot hit a point where $\lvert x - y \rvert \geq 2$ before (5,3). The resulting grid will look like this, where the bottom-left corner is (0,0) and the top-right corner is (4,3). We only need to look at (4,3) because we can't reach (5,3) from (5,2). X | X | 4 | 8 | 8 X | 2 | 4 | 4 | X 1 | 2 | 2 | X | X 1 | 1 | X | X | X
24	A	There are $\binom{6}{2} = 15$ possible couples, each with probability $\frac{1}{5} \cdot \frac{1}{5} = \frac{1}{25}$. The expected number of couples is $\frac{15}{25} = \frac{3}{5}$.
25	B	There are 2 cases 1. The new person becomes a couple There are 6 possible couples, each with probability $\frac{2}{7} \cdot \frac{1}{6}$. The expected number of couples is $\frac{2}{7}$. 2. A couple without the new person There are 15 possible couples, each with probability $\frac{1}{7} \cdot \frac{1}{7}$. The expected number of couples is $\frac{15}{49}$ Total $= \frac{2}{7} + \frac{15}{49} = \frac{29}{49}$.
26	C	If we pick 4 points that form a non-degenerate convex quadrilateral, the intersection of this quadrilateral's diagonals will be an interior intersection. There are $\binom{12}{4} = 495$ ways to choose 4 points. We now remove all the bad selections, which are when 3 or more points

		are collinear (Note that we can't pick 4 points that gives a non-degenerate concave quadrilateral). We do this with two cases. Case 1: exactly three points are collinear. For each side, we pick 3 points to be collinear. Then we pick a point not on the side as the 4th point. This gives $$8\binom{4}{3}+7\binom{5}{3}+6\binom{6}{3}=222.$$ Case 2: all 4 points are collinear. This case is straightforward. We compute $$\binom{4}{4}+\binom{5}{4}+\binom{6}{4}=21.$$ Putting everything together, we get $495-222-21=252$.
27	A	We can simplify the expression into $$\begin{aligned}&\equiv 12!\left(\frac{1}{1\cdot(-1)}+\frac{1}{2\cdot(-2)}+\frac{1}{3\cdot(-3)}+\frac{1}{4\cdot(-4)}+\frac{1}{6\cdot(-6)}\right)\\&\equiv 12!\,(-1-2^{-2}-3^{-2}-4^{-2}-6^{-2})\\&\equiv(-1)(-1-2^{-2}-3^{-2}-4^{-2}-6^{-2})\\&\equiv 1+2^{-2}+3^{-2}+4^{-2}+6^{-2}\ (mod\ 13),\end{aligned}$$ where we used Wilson's theorem to get $12!\equiv -1$ (mod 13). Now, using the observation that $1\equiv -12$ (mod 13), the expression above is equivalent to $$1+(-6)^2+(-4)^2+(-3)^2+(-2)^2\equiv 1+36+16+9+4\equiv 1\ (mod\ 13).$$
28	D	$\frac{1}{N}=\frac{1}{10^{50}}\cdot\frac{25}{(10^6+1)(10^4+1)}$. Since $\gcd(10,(10^6+1)(10^4+1))=1$, the non-repeating part has length 50. Now the repeating part is the smallest positive integer n that satisfies $(10^6+1)(10^4+1)\mid 10^n-1$. Also, since $\gcd(10^4+1, 10^6+1)=1$, we can split the equation into $10^6+1 \mid 10^n-1$ and $10^4+1 \mid 10^n-1$. Case 1. $10^4+1 \mid 10^n-1$ Noticing that $10^8-1\equiv 0$ (mod 10^4+1), $10^n-1\equiv 10^{n\ (mod\ 8)}-1\equiv 0$ (mod 10^4+1). Therefore, $n=0$ (mod 8) Case 2. $10^6+1 \mid 10^n-1$ Similarly, $10^n-1=10^{n\ (mod\ 12)}-1=0$ (mod 10^6+1) Therefore, $n=0$ (mod 12) The smallest multiple of 8 and 12 is 24.
29	B	Let $f(a,b)=\frac{1}{a^2b^2}$. The summation we are solving for can be written as $\sum_{a\le b} f(a,b)$. We can split this up into $\sum_{a<b} f(a,b)+\sum_{a=b} f(a,b)=\sum_{a<b} f(a,b)+\frac{\pi^4}{90}$. Let $\sum_{a<b} f(a,b)=I$. Due to symmetry (the fact that $f(a,b)=f(b,a)$), $\sum_{a\ne b} f(a,b)=2I$. Thus $\sum_{a,b} f(a,b)=2I+\frac{\pi^4}{90}$. Furthermore $\sum_{a,b} f(a,b)=\sum_{a=1}^{\infty}\sum_{b=1}^{\infty}\frac{1}{a^2b^2}=\sum_{a=1}^{\infty}\frac{1}{a^2}\cdot\sum_{b=1}^{\infty}\frac{1}{b^2}=\frac{\pi^4}{36}$. Therefore $2I+\frac{\pi^4}{90}=\frac{\pi^4}{36}\rightarrow I=\frac{3\pi^4}{360}$. The expression we are solving is $\frac{3\pi^4}{360}+\frac{4\pi^4}{360}=\frac{7\pi^4}{360}$.
30	E	$10\le\frac{n(n+1)}{2}<100$ yields $4\le n\le 13$ AND $-14\le n<-5$.