1. D All of the answers are continuous quantities (they can take on arbitrarily precise values in whichever unit they are measured in) except for the number of permutations of a sequence of letters, which must be a natural number.
2. B For every multiple of 15, it appears 4 times; for every multiple of 3 (and not 5), it appears 2 times; for every multiple of 5 (and not 3), it appears 2 times. We can calculate using the inclusion-exclusion principle, or intentionally double-count by counting multiples of 3 (which might also be multiples of 5) and multiples of 5 (which might also be multiples of 3): $2\left\lfloor\frac{100}{3}\right\rfloor + 2\left\lfloor\frac{100}{5}\right\rfloor = 106$.
3. C The number of lines with alphabetic characters is equal to the number of multiples of 3 or 5 between 1 and 100 inclusive, which, using the principle of inclusion-exclusion, is $\left\lfloor\frac{100}{3}\right\rfloor + \left\lfloor\frac{100}{5}\right\rfloor - \left\lfloor\frac{100}{15}\right\rfloor = 47$. The number of lines that do NOT contain alphabetic characters is $100 - 47 = 53$.
4. B An injective function is also known as a one-to-one function: if $f(a) = f(b)$, then $a = b$. II and IV are clearly injective as they are monotonic on the domain. I is clearly not injective, since $f(2019) = f(20201)$. For III, $\sin(x)$ is periodic with period of 2π. This means that if $f(a) = f(b)$, then $a = b + 2\pi k$ for some integer k. This can be rearranged to $2\pi = \frac{a-b}{k}$. However, since 2π is irrational, it can only be achieved for integer values of a, b, k when $a = b$, so III is injective.
5. E Note that $20240 = 2^4 \cdot 5 \cdot 11 \cdot 23$. 8, 40, and 880 are all factors - verify with prime factorization, so they do not belong in A^C. $1024 = 2^{10}$, so it requires 11 digits in base 2, so it does not belong in B.
6. B The cardinality of $A \cap B$ is equivalent the number of factors 20240 less than 1024. 20240 has $(4+1)(1+1)^3 = 40$ factors. Note that $\frac{20240}{20} = 1012$, which is just under 1024. So $\frac{20240}{k} \notin A$ for $k = 1, 2, 4, 5, 8, 10, 11, 16$. Thus, there are 32 elements in $A \cap B$.
7. D The negation of a universally quantified statement is the negative of an existentially qualified statement; that is, $\neg(\forall x. P(x)) = \exists x. \neg P(x)$, for some statement $P(x)$. In symbols, if we take the given statement to be $P(x)$, then the negation is that there exists ($\exists$) some cat (x) that is not devious ($\neg P(x)$).
8. A This can be rewritten in a more symbolically friendly form as "For all pots P: (there exists a lid L: (such that P and L match.))" Note that this is not the same as "There exists a lid L: (such that for all pots P: (P and L match.))" The former implies that any pot I find has a matching lid; the latter implies that there is at least one particular lid so special that every pot I find will match it.
9. D First, rewrite the implications into standard ands and ors, we have $a \wedge (a \to b) \wedge (b \to c) = a \wedge (\neg a \vee b) \wedge (\neg b \vee c)$. Apply the distributive property, we have $((a \wedge \neg a) \vee (a \wedge b)) \wedge (\neg b \vee c) = (a \wedge b) \wedge (\neg b \vee c) = (a \wedge b \wedge \neg b) \vee (a \wedge b \wedge c) = (a \wedge b \wedge c)$.
10. E The given Boolean expression is in product of sum form, and the answer choices are in sum of product form. It is logical to build a truth table in form of a Karnaugh map, then simplify from there

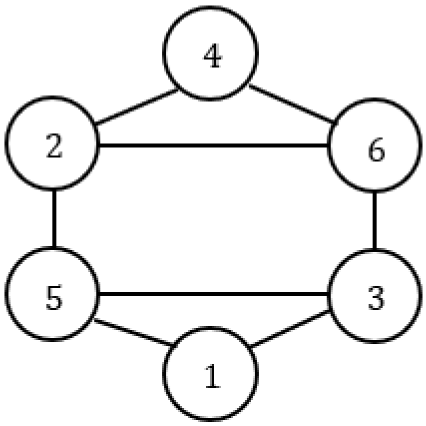
$a\backslash bc$	00	01	11	10
0	1	0	0	0
1	1	1	1	1

Grouping the first column and the second row, we have $a \vee (\neg b \wedge \neg c)$

11. A $\begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$ is straightforwardly verified to have all the correct entries. Note the symmetry about the main diagonal.

12. D A key insight to make is that the number of length-2 paths from i to j is the sum of the number of length-1 paths from i to k and from k to j for all vertices k. However, we already know these quantities as M_{ik}, M_{kj} respectively, so $N_{ij} = \sum_k M_{ik}M_{kj} \to N = M^2$. All entries can also be manually verified.

13. E The crossing number is 0. A rearrangement of vertices while retaining all edges (also called an isomorphism) that achieves this minimum is shown here:



14. C Given $2a + 3b \equiv 0 \ (mod\ 13)$, we have $2a \equiv -3b \equiv 10b$, or $a \equiv 5b$. Substituting $5b$ for a into each of I through V, it can be seen that I, III, and V are multiples of 13.

15. B The question is asking for 12^{2023} mod 100. Since 12 is not relatively prime to 100, we will consider the question mod 4 and mod 25. 12^{2023} is clearly 0 mod 4. $\phi(25) = 20$, so $12^{2023} \equiv 12^3 \equiv 1728$, or 3 mod 25. So the last two digits are 28.

16. C We can form a rectangle by choosing any two distinct horizontal lines and any two distinct vertical lines. Each has $\binom{9}{2} = 36$ ways of doing so, for a total of $36^2 = 1296$. Of those, there are 8^2 1×1 squares, 7^2 2×2 squares, ..., 1^2 8×8 squares. So the number of non-square rectangles is $1296 - \frac{8 \cdot 9 \cdot 17}{6} = 1296 - 204 = 1092$.

17. C This is equal to the number of nonempty subsets of four distinct elements. For each element (muffin), the subset (Michelle's choice) can either contain it (she takes it) or not, for a total of 2^4 subsets. Subtracting one for the empty set (where Michelle takes no muffins at all), the answer is 15.

18. E This is equal to the number of nonnegative integer solutions to $a + b + c = 6$. Using the stars and bars (or balls and urns) method, this is equivalent to the number of permutations of a string with six stars and two bars between them, which is $\binom{6+2}{2} = 28$.

19. C If $N = 1, 2,$ or 3, then the player who goes first has a winning strategy (to take all the Cheetos). If $N = 4$, then the player who goes second has a winning strategy (the first player is forced to leave 1, 2, or 3 Cheetos remaining). Similarly, if $N = 5, 6,$ or 7, then the player who goes first has a winning strategy (take $N - 4$ Cheetos to leave 4, and then this is the same as if they went second in a game with $N = 4$). If $N = 8$, then the player who goes second has a winning strategy (regardless of however many the first

player takes on the first turn, leave 4 after the second player's turn, and then this is the same as if the second player went second in a game with $N = 4$). The generalization of this is that if $N \equiv 1,2,3 \ mod\ 4$, then the player who goes first has a winning strategy: if we let N_i be the number of Cheetos left after turn i, take exactly as many Cheetos required such that $N_1 \equiv 0 \ mod\ 4$, and then the player who goes second will be forced to make $N_2 \equiv 1,2,3 \ mod\ 4$, which means the first player can repeat until $N_t = 0$.

20. A The expected value of the triangle's area is $E\left[\frac{1}{2}ab\right] = \frac{1}{2}E[a]E[b]$, where we can split the expectation because the two discrete random variables a and b are independent. $E[a] = \sum_{i=0}^{\infty} i\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^i$, and $E[b] = \sum_{i=0}^{\infty} i\left(\frac{3}{7}\right)\left(\frac{4}{7}\right)^i$. In order to solve this, we need a preliminary result, $\sum_{i=0}^{\infty} ix^i$ for some value x. This is equivalent to

$$\begin{aligned}
&x^1 + 2x^2 + 3x^3 + 4x^4 \ldots \\
&= (x^1 + x^2 + x^3 + x^4 + \cdots) \\
&\quad + (x^2 + x^3 + x^4 + \cdots) \\
&\quad + (x^3 + x^4 + \cdots) \\
&\quad + (x^4 + \cdots) \\
&\quad + \cdots \\
&= \left(\frac{x}{1-x}\right) + \left(\frac{x^2}{1-x}\right) + \left(\frac{x^3}{1-x}\right) + \cdots \\
&= \left(\frac{1}{1-x}\right)(x + x^2 + x^3 + \cdots) \\
&= \frac{x}{(1-x)^2}
\end{aligned}$$

(A faster and completely different method of finding the value of the summation is also presented in most second-year calculus curricula.) Note that the infinite series steps only converge when $|x| < 1$, which is true in both of the summations in question. Following this formula, $E[a] = 2$ and $E[b] = \frac{4}{3}$, so the expected area is $\left(\frac{1}{2}\right)(2)\left(\frac{4}{3}\right) = \frac{4}{3}$.

21. B Let the "search space" be the set of numbers that Nora's secret number could be. If Dale plays optimally, every guess cuts the search space in half. If the size of the search space is odd, the size is rounded down when dividing in half. (Example: when guessing a number from 1-5 inclusive, the search space size shrinks from 5 to $\left\lfloor\frac{5}{2}\right\rfloor = 2$ when guessing the midpoint 3, if the guesser hasn't already won.) She has guaranteed a win when the search space size is 0. After successive turns, the search space contains 50, 25, 12, 6, 3, 1, 0 numbers, which is seven guesses.

22. A It is true that the graph has a cycle (e.g. 1, 2, 3 form a cycle), it is true that the graph is undirected, it is true that the graph is not a tree (because it has cycles), it is not true that the graph is complete because there are not edges between every possible pair of vertices.

23. C The only vertices adjacent to 1 are 2 and 3. We will consider paths starting with 1-2. 2 can connect to 1, 3, 4, 5. Paths going to 1 and 4 must return to 2 then to 3. 5 can go to 2 or 6, then to 3. Path to 3 can leave 3 to 1, 2, 5, or 6 then return. This makes for 8 paths starting with 1-2. For paths starting with 1-3, we are now at 3, and need two more steps to return to 3, which means 3-A-B-3 must form a triangle of paths. There are 3 triangles with 3 as a vertex: 1-2-3, 2-3-5, 3-5-6. Each triangle can be traversed in clockwise or counterclockwise order, making for 6 paths. In total, there are 14 paths.

24. E Both depth first and breadth first searches follow this pattern: from the current node n, add all neighbors of n to a data structure N (if they are not already visited or added to

N) of nodes to visit in the future. Remove a node from N to visit next and repeat by visiting that node (i.e. set it to the current node n), until n equals the target.

The difference is that in depth first search, N is a stack (last-in-first-out) and in breadth first search, N is a queue (first-in-first-out).

We trace the BFS below. Note that nodes are added to the queue back and removed from the queue front. This means we need to add neighbors increasing order of number, in accordance with the conditions in the question.

$$n = current\ node, Q = (queue\ front)[\dots,\dots](queue\ back)$$
$$n = 1, Q = [2,3]$$
$$n = 2, Q = [3,4,5]$$
$$n = 3, Q = [4,5,6]$$
$$n = 4, Q = [5,6,7,8]$$
$$n = 5, Q = [6,7,8,9]$$
$$n = 6, Q = [7,8,9,10]$$
$$n = 7, Q = [8,9,10]$$
$$n = 8, Q = [9,10,11]$$
$$n = 9, Q = [10,11,12,13]$$
$$n = 10, Q = [11,12,13,14]$$
$$n = 11, Q = [12,13,14,15]$$
$$n = 12, Q = [13,14,15]$$
$$n = 13, Q = [14,15]$$
$$n = 14, Q = [15,16]$$
$$n = 15, Q = [16]$$
$$n = 16, Q = [\ ]$$

16 nodes are visited, including 1 and 16.

25. C Every edge is part of a cycle, and hence will not disconnect the graph if removed, except for the two edges between 4—7 and between 9—13.

26. A A nonplanar graph has a subgraph that is homeomorphic to K_5 or $K_{3,3}$. It's clear that since vertices 7 and 13 have degree of 1, they can be completely neglected (effectively removed from the graph). We can simplify the rest of the problem by smoothing out edges that pass through vertices with degree 2. Edges 6-10-14-16-15 can be smoothed into 6-15, 5-9-12-15 into 5-15, 2-4-8-11-15 into 2-15. Note that 2-1-3 should not be smoothed since 2-3 is an existing edge.

- First, consider K_5, there are 8 edges connecting nodes 2, 3, 5, 6, and 15, so 2 additional edges are needed. Specifically, 2-6 and 3-15.
- Second, consider $K_{3,3}$, consider 2 sets $\{1, 5, 15\}$ and $\{2, 3, 6\}$. If edges 3-15 and 1-6 are added, then the resulting graph is $K_{3,3}$.

In either case, 2 edges need to be added.

27. B Each node in a binary tree has 2 children, so each level has 2 times more nodes than the previous, starting from the first: level 1 has 1, level 2 has 2, level 3 has 4, level 4 has 8… the pattern is that level n has 2^{n-1} nodes.

28. A We present a divide-and-conquer approach by incrementally adding dominoes in one of two ways. (It's straightforward to see that there are no other ways to incrementally add dominoes, and that incrementally adding dominoes in either of these ways indeed produces all possible outcomes.) Note that when incrementally tiling a $2 \times n$

checkerboard for $n > 2$, we can either put down a vertical domino in the first column, in which case we are left with a $2 \times (n-1)$ checkerboard, or we can put down two horizontal dominoes stacked atop each other, in which case we are left with a $2 \times (n-2)$ checkerboard. If $f(n) =$ the number of ways to tile a $2 \times n$ checkerboard, then $f(n) = f(n-1) + f(n-2)$ by the logic above. It's also clear that $f(1) = 1$ and $f(2) = 2$. Following this Fibonacci-like sequence, we get $f(10) = 89$.

29. D Expanding the recurrence,

$$T(n) = 8T\left(\frac{n}{2}\right) + n^3$$
$$T(n) = 8\left(8T\left(\frac{n}{2^2}\right) + \left(\frac{n}{2^1}\right)^3\right) + n^3$$
$$T(n) = 8\left(8\left(8T\left(\frac{n}{2^3}\right) + \left(\frac{n}{2^2}\right)^3\right) + \left(\frac{n}{2^1}\right)^3\right) + n^3$$
$$T(n) = 8^{\log_2 n} T\left(\frac{n}{n^{\log_2 n}}\right) + \sum_{i=0}^{(\log_2 n)-1} 8^i \left(\frac{n}{2^i}\right)^3$$
$$T(n) = n^3 T(1) + n^3 \sum_{i=0}^{(\log_2 n)-1} 1$$
$$T(n) = 2n^3 + n^3 \log_2 n$$

So $0 + 2 + 3 + 1 + 3 = 9$

30. E Since 9 and 15 are multiples of 3, we will consider those 2 packages first, which can only result in multiples of 3. The number of purchasable nuggets is $9a + 15b = 3(3a + 5b)$, so the maximum unpurchasable multiple of 3 using 9 and 15 is $3(3 \cdot 5 - 3 - 5) = 21$. More specifically, the unpurchasable multiples of 3 are 3, 6, 12, and 21.

Next, we will include packs of 8. The purchasable nugget count is now $8x + 3y$, where x, y are nonnegative integers and $y \neq 1, 2, 4, 7$. We will consider $8x + 3y$ modulo 3:

- 0 mod 3: $8x$ can be neglected, so the largest unpurchasable is 21 by above.
- 1 mod 3: smallest purchasable is $8 \cdot 2 = 16$a, so the largest unpurchasable is $13 + 21 = 34$.
- 2 mod 3: smallest purchasable is 8, so the largest unpurchasable is $5 + 21 = 26$.

Overall, the largest un-purchasable nugget quantity is 34.