1. D $2023 = 7 \cdot 17^2$, so the sum of its factors is $(1+7)(1+17+289) = 8 \cdot 307 = 2456$.
2. B $\varphi(2023) = \varphi(7 \cdot 17^2) = \varphi(7)\varphi(17^2) = 6 \cdot 17\varphi(17) = 6 \cdot 17 \cdot 16 = 1632$.
3. C It is easy to confirm 461 is prime, so $\varphi(461) = \varphi(2 \cdot 461) = 460$. Additionally, 10 and 46 are both 1 less than primes, so $\varphi(11 \cdot 47) = \varphi(507) = \varphi(1114) = 460$ for a total of 4 values of n.
4. D By the Chinese Remainder Theorem, there is a unique value of A mod $5 \cdot 7 \cdot 11 = 385$. Investigation of the first pair yields $A \equiv 12$ mod 35. Further investigation shows $A \equiv 257$ mod 385. The smallest solution to $2023 < A = 385n + 257$ is for $n = 5$, giving $A = 2182$. This is 16 mod 19.
5. D Note that $27 = 5^2 + 2$, so $27 \cdot 25$ is 1 greater than a perfect square. $A = 26$ and $B = 5$. $AB = 130$, which is 16 mod 19.
6. B Bezout's theorem states that the integers of the form $12x + 16y$ are the ones of the form $kGCD(12,16) = 4k$. The sum of the multiples of 4 less than 100 is equal to $4\sum_{i=1}^{25} i = 2 \cdot 25 \cdot 26 = 1300$.
7. B In base b, if the sum of the digits of an integer is divisible by $b-1$, then the number is divisible by $b-1$; furthermore, the sum of digits is equivalent to the number modulo $b-1$. Thus, we are looking for a choice whose sum of digits is equivalent to 0 or 9 mod 18. By inspection, $B + I + G + H + E + A + D = 11 + 18 + 16 + 17 + 14 + 10 + 13 = 99$, which satisfies the condition. For completeness's sake, $BIGHEAD_{19} = 564281469 = 9 \cdot 62697941$.
8. D It is easy to confirm that 3 is not a quadratic residue of $p = 5$. Choices A through C are the Law of Quadratic Reciprocity and two of its supplements (D gives the negative value of the Legendre symbol).
9. A $\left(\frac{50}{83}\right) = \left(\frac{2}{83}\right)\left(\frac{5^2}{83}\right) = \left(\frac{2}{83}\right) = -1$, since $83 \equiv 3$ mod 8.
   $\left(\frac{29}{37}\right) = \left(\frac{-8}{37}\right) = \left(\frac{-1}{37}\right)\left(\frac{2}{37}\right)\left(\frac{2^2}{37}\right) = 1 \cdot -1 = -1$, since $37 \equiv -3$ mod $8 \equiv 1$ mod 4.
   $\left(\frac{2023}{19}\right) = \left(\frac{9}{19}\right) = \left(\frac{3^2}{19}\right) = 1$.
   $-9 - 3 + 1 = -11$.
10. D The solution to $3x \equiv 1$ mod 26 is $x \equiv 9$ mod 26. Thus, the inverse of $3n + 2$ modulo 26 is $9(n-2) = 9n - 18$. As a set, this is the transformation $\{18,13,10,7\} \to \{144,99,72,45\} \equiv \{14,21,20,19\}$ mod 26, representing the word NUTS. $14 + 21 + 20 + 19 = 74$.
11. C $12! = 2^{10}3^5 5^2 7^1 11^1$. The number of positive integer factors this has is equal to $11 \cdot 6 \cdot 3 \cdot 2 \cdot 2 = 792$
12. A $\sum_{n=1}^{\infty}\left\lfloor\frac{2023}{2^n}\right\rfloor = 1011 + 505 + 252 + 126 + 63 + 31 + 16 + 8 + 4 + 2 + 1 = 2019$. $\sum_{n=1}^{\infty}\left\lfloor\frac{2023}{3^n}\right\rfloor = 674 + 224 + 74 + 24 + 8 + 2 + 1 = 1007$. The largest factor of 8 that divides $2023!$ is $\left\lfloor\frac{2019}{3}\right\rfloor = 673$. The largest factor of 9 that divides $2023!$ is $\left\lfloor\frac{1007}{2}\right\rfloor = 503$.
13. A The number of digits is $1 + \lfloor\log_{10} 5^{23}\rfloor = 1 + \lfloor 23\log_{10} 5\rfloor \approx 1 + \lfloor 23 \cdot 0.699\rfloor = 1 + 16 = 17$.

14. B This is equivalent to finding the number of solutions in the whole numbers to $3x + 14y + 15z = 420 - 3 - 14 - 15 = 388$. $3x + 14y + 15z = 14y + 3(x + 5z) = 14y + 3n$. The cases this makes are $3n = \{360,318,276,234,192,150,108,66,24\}$, so $x + 5z = \{120,106,92,78,64,50,36,22,8\}$. The number of solutions in each case is 1 plus the floor of each case divided by 5, which is $\{25,22,19,16,13,11,8,5,2\}$ for a total of 121 solutions.

15. C By Fermat's Little Theorem, $19^{22} \equiv 1$ mod 23. $2023 \equiv -1$ mod 23, so we are looking for the inverse of 19 (or -4) modulo 23. $-68 = 1 + (-69)$, so this inverse is 17, and indeed, $19 \cdot 17 = 323 \equiv 1$ mod 23.

16. B In the sum of factors function, 31^1 in the factorization would contribute $1 + 31 = 32$ to the product. Note that the sum of the factors of $16 = \frac{32}{2}$ is $1 + 2 + 4 + 8 + 16 = 31$. Therefore, $31 \cdot 16 = 496$ is a perfect number, and there are no smaller ones divisible by 31. $4 + 9 + 6 = 19$. Generally, if $2^p - 1$ is prime, then $2^{p-1}(2^p - 1)$ is perfect.

17. A The recursion is homogeneous, and the characteristic polynomial is $x^2 - 5x + 6 = 0$. The roots of this are 2 and 3, so $a_n = p2^n + q3^n$. Solving the system of equations $\{p + q = -7, 2p + 3q = -12\}$ gives the solution $p = -9$, $q = 2$. Matching this with the given expression yields $A = 2$, $B = 3$, $C = 9$, $D = 2$. $\begin{vmatrix} 2 & 3 \\ 9 & 2 \end{vmatrix} = 4 - 27 = -23$.

18. D Let the exponents of 2, 3, and 5 in the prime factorization of N be a, b, and c in some order. We have the congruencies $a \equiv \begin{matrix} 1 \text{ mod } 2 \\ 0 \text{ mod } 3 \\ 0 \text{ mod } 5 \end{matrix}$, $b \equiv \begin{matrix} 0 \text{ mod } 2 \\ 2 \text{ mod } 3 \\ 0 \text{ mod } 5 \end{matrix}$, $c \equiv \begin{matrix} 0 \text{ mod } 2 \\ 0 \text{ mod } 3 \\ 4 \text{ mod } 5 \end{matrix}$. The primitive solutions to these, respectively, are 15, 20, and 24. Thus, $N = 2^{24}3^{20}5^{15}$. Since $2^3 \equiv 1$ mod 7, $3^2 \equiv 2$ mod 7, and $5^3 \equiv -1$ mod 7, this is equivalent to $1^8 2^{10}(-1)^5$ mod 7, which simplifies to -2^{10} and subsequently $-2 \equiv 5$ mod 7.

19. B 219 is divisible by 3, so the mystery product is divisible by 9 and its sum of digits is a multiple of 9. The given digits sum to 49, so a 5 is missing.

20. C The given product is $2^{48}3^{31}5^7 7^8$, which has $\left\lfloor 1 + \frac{48}{3} \right\rfloor \left\lfloor 1 + \frac{31}{3} \right\rfloor \left\lfloor 1 + \frac{7}{3} \right\rfloor \left\lfloor 1 + \frac{8}{3} \right\rfloor = 17 \cdot 11 \cdot 3 \cdot 3 = 1683$ cube factors.

21. C The denominator factors to $(2n + 5)(n - 3)$. The odd factors of 336 are $-21, -7, -3, -1, 1, 3, 7$, and 21. Setting $2n + 5$ equal to these gives $n = \{-13, -6, -4, -3, -2, -1, 1, 8\}$, corresponding to $n - 3 = \{-16, -9, -7, -6, -5, -4, -2, 5\}$. 5 of these are factors of 336.

22. A $x = 480n + 80 = 120(4n) + 80$. Counterexamples to II and III are $x = 80$ and $x = 560$ respectively. Only I is true.

23. C By Euler's theorem, $a^{\varphi(2023)} \equiv 1$ mod 2023. It has previously been established that $\varphi(2023) = 1632$, so $2^{1632} \equiv 1$ mod 2023. $2^{1632} = 4^{816}$. We are looking for 4^{4096} mod 2023. By taking $(4^{816})^5$, we have $4^{4080} \equiv 1$ mod 2023 and would only need to find 4^{16} mod 2023. $4^{16} = 2^{32}$, so $N = 5$.

24. B The last digit of a fifth power is identical to the last digit of the integer taken to the fifth power to obtain it. Noting that $200^5 = 320{,}000{,}000{,}000$, $N = 2A7$ for some digit A. $N \equiv 1$ mod 9. The only value of A such that $2A7^5 \equiv 1$ mod 9 is 1. $2 + 1 + 7 = 10$

25. B The iterations of $\{b\}$ are shown here.
$\{5,5,5\}$
$\{5,5,6,30\}$

$\{5,6,6,30,30\}$
$\{5,6,7,30,30,42\}$
$\{5,6,7,30,31,42,930\}$
The sum of these numbers is 1051. $\frac{3}{5}=\frac{1}{2}+\frac{1}{10}$ can be found with other algorithms.

26. E $12^8=2^{16}3^8$, $14^{23}=2^{23}7^{23}$, and $21^{12}=3^{12}7^{12}$. The number of factors of these numbers individually are $17\cdot 9=153$, $24\cdot 24=576$, and $13\cdot 13=169$ respectively, for a total of 898 factors. However, by the Principal of Inclusion and Exclusion, there is double counting. $GCF(12^8,14^{23})=2^{16}$, $GCF(14^{23},21^{12})=7^{12}$, and $GCF(21^{12},12^8)=3^8$. $17+13+9=39$ factors must be removed to bring the total to 859. However, also by PIE, there is triple counting, as the factor 1 was removed thrice and must be re-counted to make for 860 unique divisors.

27. A By inspection, $17=1^3+4^2=2^3+3^2$ is the smallest. $1+7=8$.

28. E $2023=2187-243+81-3+1$, or +0−+00−+ in balanced ternary.

29. C WLOG, let $a<b<c<d$. Either $a+b+c=d$ or $a+b+c=2d$. Consider the first case. $a|b+c+d$, so $a,b,c|2d$. Let $2d=ax=by=cz$, where $2<z<y<x$. Then $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{2}$. If $z=3$, then $\frac{1}{x}+\frac{1}{y}=\frac{1}{6}$, giving the solutions $(x,y)=\{(42,7),(24,8),(18,9),(15,10)\}$. This generates the solution sets $\{(k,6k,14k,21k),(k,3k,8k,12k),(k,2k,6k,9k),(2k,3k,10k,15k)\}$. If $z=4$, then $\frac{1}{x}+\frac{1}{y}=\frac{1}{4}$, giving the solutions $(x,y)=\{(20,5),(12,6)\}$ and the solution sets $\{k,4k,5k,10k\}$ and $\{k,2k,3k,6k\}$. There are no solutions for $z=5$, and $z\geq 6$ would make it impossible to find denominators that sum to $\frac{1}{2}$. If $a+b+c=2d$, then $a,b,c|3d$ and $3d=ax=by=cz$ for $3<z<y<x$. $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{2}{3}$. However, $x\geq 4$, $y\geq 5$, and $z\geq 6$, so the maximum value of $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{37}{60}<\frac{2}{3}$ and there are no solutions. $k=2$ gives 5 solutions from the sets obtained, and $k=1$ gives 3 solutions for a total of 8. For completeness's sake, the sets are {1,2,3,6}, {1,2,6,9}, {2,3,10,15}, {2,4,6,12}, {2,4,12,18}, {2,6,16,24}, {2,8,10,20}, and {2,12,28,42}.

30. A If the number has at least four distinct prime divisors, then its value is at least $2^{14}3^15^17^1>9999$. If it has three distinct prime divisors, these must be 2, 3, and 5. The possible values are $2^83^15^1=3840$, $2^73^25^1=5760$, $2^63^35^1=8640$, and $2^73^15^2=9600$. If it has two distinct prime divisors, 7 is a possible prime factor. The possible values are $2^45^3=2000$, $2^35^4=5000$, $2^87^1=1792$, and $2^77^2=6272$. If it has one prime factor, the possible value is $5^5=3125$. This is a total of 9 solutions.